National Library of Canada

Acquisitions and Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale du Canada

Direction des acquisitions et des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file *Votre référence*

Our file *Notre référence*

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

Canada

UNIVERSITY OF ALBERTA

POLYNOMIAL APPROXIMATION

By

DECHANG JIANG

A thesis submitted to the FACULTY OF GRADUATE STUDIES AND RESEARCH in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

Department of Mathematics

Edmonton, Alberta

Spring, 1993

National Library of Canada
Acquisitions and Bibliographic Services Branch
395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale du Canada
Direction des acquisitions et des services bibliographiques
395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file *Votre référence*

Our file *Notre référence*

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-82169-8

Canada

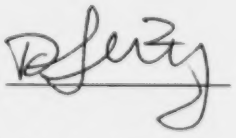
UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR:	Dechang Jiang
TITLE OF THESIS :	Polynomial Approximation
DEGREE :	Doctor of Philosophy
YEAR THIS DEGREE GRANTED:	1993

Permission is hereby granted to the University of Alberta library to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves all other publication and other rights in association with the copyright in the thesis, and except as hereinbefore provided neither the thesis nor any substantial portion thereof may be printed or otherwise reproduced in any material form whatever without the author's prior written permission.

Signature 

Department of Mathematics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

Date: April, 13, 1993

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled POLYNOMIAL APPROXIMATION submitted by DECHANG JIANG in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

Z. Ditzian, *Supervisor*

A. Z. Capri

R. Q. Jia

R. Poliquin

S. D. Riemenschneider

A. Sharma

V. Zizler

P. Nevai, *External examiner*

Date: April, 13. 1993

To Lin, My Family,

and

Wuhan—the city I came from.

献给琳，我的家　也献给我的家乡－武汉

ABSTRACT

This thesis focuses on studies of the polynomial approximation of real continuous functions on finite intervals. The well-known Timan-type pointwise direct estimates and the uniform norm direct estimates by Ditzian and Totik are unified. The corresponding inverse estimates are unified as well. When the moduli of continuity and smoothness of the second order are used to measure the approximation rates, the approximating polynomials can be chosen to interpolate the approximated function at end points of the interval and moreover, the polynomials can also be monotone, convex or both whenever the function is monotone, convex or both. If the approximated function is differentiable, the function and its derivatives can be simultaneously approximated by polynomials and their derivatives. Moreover, the polynomials can be required to interpolate the function at given points.

ACKNOWLEDGEMENT

With the sincerest gratefulness, I wish to thank my supervisor, Professor Z. Ditzian, for his introduction to this research field, his guidance, and his concern during my Ph.D. research.

It is with profound appreciation that I thank Professor A. Sharma for his unforgettable kindness and help. It is a great pleasure to thank Professor S. D. Riemenschneider for his invaluable advice and interest in my research. I would also like to express my warm thanks to Professor R. Q. Jia for his consistent encouragement and supportive advice.

Thanks also go to Professor D. Leviatan as working with him in the summer of 1991 was a learning experience. There were numerous helps or concerns from teachers, friends, the Department, and my family; I wish to thank all of them.

TABLE OF CONTENTS

Abstract

Chapter 1. Introduction 1

1.1. Preamble 1

1.2. Outline of the thesis 3

References 7

Chapter 2. Approximation of Functions

by Polynomials in $C[-1,1]$ 9

2.1. Introduction 9

2.2. The direct estimate 11

2.3. Construction of the auxiliary function G_n 16

2.4. Estimate on derivatives of polynomials 21

2.5. The converse result 25

2.6. The derivatives of the approximating polynomial 30

2.7. Remarks 36

References 38

Chapter 3. Shape Preserving
Polynomial Approximation in $C[-1,1]$ 40
3.1. Introduction and the main result 40
3.2. Proof of the main result 43
3.3. A Jackson-type inequality
for an intermediate operator 47
3.4. The Jackson-type inequality for $L_n(f,x)$ 49
References 55

Chapter 4. Simultaneous Polynomial Approximation 56
4.1. Introduction 56
4.2. Some preliminary Lemmas 59
4.3. The main result 66
References 74

Chapter 5. Approximation by Polynomials
with Interpolatory Conditions 77
5.1. Introduction 77
5.2. Some Lemmas 79
5.3. Proof of The Theorem 81
References 87

CHAPTER I.

INTRODUCTION

1.1. Preamble

Polynomial approximation has long been one of the active fields in approximation theory. Starting in 1951, Timan [13] and later on, Dzjadyk, Freud and Brudnyĭ gave pointwise estimates for polynomial approximation of continuous functions (see Lorentz [11] and Timan [14]). After the papers of Timan and Brudnyĭ, considerable effort was devoted to polynomial approximation rates in L_p and to polynomial approximation using norm estimates rather than pointwise estimates. These investigations led to the search for new types of moduli of smoothness. In 1980, using his moduli, K. G. Ivanov solved the problem of characterization of the best polynomial approximation. A simpler measure of smoothness was introduced by Z. Ditzian and V. Totik. Their moduli include and are a natural generalization of the well-known classical moduli of smoothness. With their moduli, Ditzian and Totik were able to write down a characterization of the best polynomial approximation in L_p, $1 \le p \le \infty$, (see Chapter 7 in [5]) which yields a complete analogue to the elegant theory for trigonometric polynomial approximation (see [11]).

The Timan-type pointwise estimates and the uniform norm estimates for $C[-1,1]$ given by Ditzian and Totik are different. We will give results which

contain both and so unify these theories.

Each chapter is intended as an independent paper including its own references, but the chapters are interrelated and developed in a natural sequence.

We now introduce some notations that will enable us to describe the results of the thesis in a general way. $C[a,b]$ denote the space of continuous functions with the norm

$$\|f\| \equiv \|f\|_C = \max_{a\le x\le b} |f(x)| \tag{1.1.1}$$

and then, the space of functions with s continuous derivatives can be denoted by

$$C^s[a,b] \equiv \left\{ f \in C[a,b] \,;\; f^{(i)} \in C[a,b],\ i = 1,\cdots,s \right\}. \tag{1.1.2}$$

The Ditzian-Totik moduli of smoothness are defined by

$$\omega_\psi(f,t) \equiv \omega^1_\psi(f,t), \quad \omega^r_\psi(f,t) \equiv \sup_{0<h\le t} \|\Delta^r_{h\psi} f\| \tag{1.1.3}$$

where r is a nonnegative integer and the difference operator with the step-weight function ψ is defined as

$$\Delta^r_{h\psi(x)} f(x) = \begin{cases} \sum_{j=0}^r (-1)^j \binom{r}{j} f(x + (j - \frac{r}{2})h\psi(x)) & \text{if } x \pm \frac{r}{2} h\psi(x) \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

The step-weight function ψ discussed in [5] should be nonnegative and satisfy certain conditions. The functions

$$\varphi(x) = \sqrt{(x-a)(b-x)} \quad \text{on} \quad [a,b], \qquad \varphi(x) = \sqrt{1-x^2} \quad \text{on} \quad [-1,1],$$

and $\varphi^{\lambda}(x)$, with $0 \leq \lambda \leq 1$, satisfy these conditions.

The modulus $\omega^{r}_{\varphi^{\lambda}}(f,t)$, $0 \leq \lambda \leq 1$, will play a central role in this thesis. Observe here that when $\lambda = 0$, it is the ordinary modulus of smoothness (see [11]), i.e., $\omega^{r}_{1}(f,t) = \omega^{r}(f,t)$.

1.2. Outline of the Thesis

The following is a brief summary of the various chapters. In each part, the main results of the chapter are presented after a brief introduction.

Chapter 2. The Timan-type theorem, which can be found in various texts (see *pp.*65-66 in [11] and *p.*262 in [14]), states that for every function $f \in C[-1,1]$ there exists a sequence $\{P_n\}$ of polynomials of degree n satisfying

$$\begin{aligned} |f(x) - P_n(x)| &\leq C(r)\omega^{r}\left(f, n^{-1}(n^{-1} + \varphi(x))\right) \\ &\equiv C(r)\omega^{r}\left(f, n^{-1}\delta_n(x)\right) \end{aligned} \tag{1.2.1}$$

where $\delta_n(x) = n^{-1} + \varphi(x)$ and $\omega^{r}(f,t)$ is the standard modulus of smoothness ($\omega^{r}_{\psi}(f,t)$ with the step-weight function $\psi \equiv 1$, see (1.1.3)).

Recently, Ditzian and Totik (see Chapter 7 in [5]) showed that a sequence $\{P_n\}$ of polynomials of degree n can be found to satisfy

$$\|f - P_n\|_C \leq C(r)\omega^{r}_{\varphi}(f, n^{-1}). \tag{1.2.2}$$

In fact the best approximation polynomials in $C[-1,1]$ will satisfy (1.2.2).

One can observe that the two results (1.2.1) and (1.2.2) are different. It will be shown in this chapter that for any $f \in C[-1,1]$ one can find a sequence of

polynomials $\{P_n\}$ of degree n satisfying

$$|f(x) - P_n(x)| \le C(r,\lambda)\omega^r_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}), \quad 0 \le \lambda \le 1. \tag{1.2.3}$$

The inequality (1.2.3) coincides with (1.2.1) when $\lambda = 0$ and with (1.2.2) when $\lambda = 1$.

On the other hand, if for $f(x) \in C[-1,1]$ and for some given λ, $0 \le \lambda \le 1$, a sequence of polynomials $\{P_n\}$ satisfies

$$|f(x) - P_n(x)| \le C\omega(n^{-1}\delta_n(x)^{1-\lambda}), \quad (\ n = 1, 2, \cdots\ )$$

with an increasing function $\omega(t)$ which satisfies

$$\omega(\mu t) \le C(\mu^r + 1)\omega(t)$$

for any $\mu, t > 0$ with C independent of μ and t, then for any $t > 0$ the following inequality holds:

$$\omega^r_{\varphi^\lambda}(f,t) \le C(r,\lambda)t^r \sum_{0<n\le t^{-1}} (n+1)^{r-1}\omega(n^{-1}).$$

This is an inverse result for (1.2.3). It includes results of Timan and others for $\lambda = 0$ and a result of Ditzian and Totik for $\lambda = 1$.

Chapter 3. When a function is monotone and/or convex, we prefer that the approximating polynomials are monotone and/or convex as well. In 1985, DeVore and Yu [4] showed that for a function $f \in C[-1,1]$ which is monotone, one can

construct a monotone polynomial P_n of degree n such that

$$|f(x) - P_n(x)| \le C\omega^2\left(f, \frac{\sqrt{1-x^2}}{n}\right), \qquad -1 \le x \le 1. \tag{1.2.4}$$

Later, in 1986, Leviatan [9] pointed out that when f is convex, so is the above constructed polynomial P_n. Leviatan [9,10] also proved that $\omega_\varphi^2(f, n^{-1})$ can replace $\omega^2(f, n^{-1}\sqrt{1-x^2})$ in (1.2.4) with P_n preserving monotonicity or convexity.

The above results can be strengthened. It will be shown in chapter 3 that for $f(x) \in C[-1,1]$, $0 \le \lambda \le 1$ and $\varphi(x) = \sqrt{1-x^2}$, there exists P_n of degree n with $P_n(\pm 1) = f(\pm 1)$ such that

$$|f(x) - P_n(x)| \le C(\lambda)\omega_{\varphi^\lambda}^2\left(f, n^{-1}\varphi(x)^{1-\lambda}\right), \ -1 \le x \le 1$$

with $C(\lambda)$ independent of n and f. Moreover, if f is monotone or convex, or both, P_n can be chosen to be monotone or convex or both, respectively. The choice of P_n may be made independent of λ. We observe that $\omega_{\varphi^\lambda}^2(f,t) \le C\omega_{\varphi^\lambda}(f,t)$ [5] (or see (5.2.7) in Chapter 5). Hence, the above result is also valid for the modulus of continuity.

Chapter 4. For functions in $C^s[-1,1]$, it has been known that the derivatives of the approximating polynomials will also approximate the derivatives of the functions up to the s-th derivative simultaneously. Here it is shown that for $f \in C^s[-1,1]$ there exists a sequence $\{P_n\}$ of polynomials of degree n such that for $0 \le \nu \le s$

$$\left|f^{(\nu)}(x) - P_n^{(\nu)}(x)\right| \le C(r,s,\lambda)\left[n^{-1}\delta_n(x)\right]^{s-\nu}\omega_{\varphi^\lambda}^r\left(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\right) \tag{1.2.5}$$

where $\delta_n(x) = n^{-1} + \varphi(x)$ and for $k \geq r$

$$\left|P_n^{(s+k)}(x)\right| \leq C(r,s,\lambda)\left[n^{-1}\delta_n(x)\right]^{-k}\omega_{\varphi^\lambda}^r\left(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\right). \tag{1.2.6}$$

When it comes to $r = 1$ and $r = 2$, we can make the above result sharper in the following manner:

$$\left|f^{(\nu)}(x) - P_n^{(\nu)}(x)\right| \leq C(r,s,\lambda)\left[n^{-1}\varphi(x)\right]^{s-\nu}\omega_{\varphi^\lambda}^\ell\left(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\right) \tag{1.2.7}$$

for $\ell = 1, 2$ and $0 \leq \nu \leq s$, and

$$\left|P_n^{(s+k)}(x)\right| \leq C(r,s,\lambda)\left[n^{-1}\delta_n(x)\right]^{-k}\omega_{\varphi^\lambda}^r\left(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\right)$$

for $k \geq \ell$.

The results are new for $\lambda > 0$. For the special case $\lambda = 0$ earlier investigations of this type seem to go back to Timan [13] in 1951. His work was extended in the sixties by Trigub [15] and Brudnyĭ [1] who obtained (1.2.5) for $\lambda = 0$. The improvement (1.2.6) for $\lambda = 0$ which consisted of replacing $\delta_n(x)$ by $\varphi(x)$ is due to Teljakovski [12], Gopengauz [8], DeVore [2] and [3], and Gonska and Hinnemann [7].

Chapter 5. The results that we have presented for Chapters 2 and 4 are about approximation of functions by polynomials without any restrictive conditions. In this chapter, we investigate the approximation of functions by the polynomials which also interpolate the functions.

Given points $\{x_1, \cdots, x_l\}$ in $[a,b]$, we will show that for any function $f \in C^s[a,b]$, there exists a sequence $\{P_n\}$ of polynomials which satisfy (1.2.5)

and (1.2.6) such that

$$P_n^{(\nu)}(x_j) = f^{(\nu)}(x_j) \qquad 1 \le j \le l, \quad 0 \le \nu \le s.$$

References

[1] Ju. A. Brudnyĭ, *Generalizations of a theorem of A. F. Timan*, Dokl. Akad. Nauk SSSR **148** (1963), 1237-1240 (Russian); Soviet Math. Dokl. **4** (1963), 244-247 (English).

[2] R. A. DeVore, *Pointwise approximation by polynomials and splines*, Theory of approximation of functions (S. B. Stečkin and S. A. Teljakovskiĭ, eds.), (proceedings of an Intern. Conf. Kaluga 1975), Nauka Moskow, 1977, pp. 132–141.

[3] R. A. DeVore, *Degree of approximation*, Approximation Theory II (G. G. Lorentz, C. K. Chui and L. L. Schumaker, eds.), (Proc. Int. Sympos. Austin TX 1976), Academic Press New York, 1976, pp. 117–161.

[4] R. A. DeVore and X. M. Yu, *Pointwise estimates for monotone polynomial approximation*, Constr. Approx. **1** (1985), 323-331.

[5] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, 1987.

[6] L. O. Ferguson, "Approximation by Polynomials with Integral Coefficients," Mathematical Surveys · Number 17, Amer. Math. Soc., Providence(1980), Rhode Island.

[7] H. Gonska and E. Hinnemann, *Punktweise abschätzungen zur approximation durch algebraische polynome*, Acta Math. Hung. **46** (1985), 143–254.

[8] I. E. Gopengauz, *A theorem of A.F. Timan on the approximation of functions by polynomials on a finite segment*, Mat. Zametki (1967) 163-172 (Russian); Math. Notes **1** (1967) 110-116 (English).

[9] D. Leviatan, *Pointwise estimates for convex polynomial approximation*, Proc.

Amer. Math. Soc. **98** (1986), 471-474.

[10] D. Leviatan, *Monotone and comonotone polynomial approximation revisited,* J. Approx. Theory **53** (1988), 1-16.

[11] G. G. Lorentz, "Approximation of Functions," Holt, Rinehart and Winston, 1966. (Reprinted by Chelsea, 1988).

[12] S. A. Teljakovskiĭ, *Two theorems on the approximation of functions by algebraic polynomials,* Mat. Sb. **70** (1966), 252–265 (Russian).

[13] A. F. Timan, *Strengthening of Jackson's theorem on the approximation of continuous functions on a finite segment of the real axis,* Dokl. Akad. Nauk SSSR **78** (1951), 17–20 (Russian).

[14] A. F. Timan, "Theory of Approximation of Functions of a Real Variable," English translation 1963, Pergamon Press, The MacMillan Co. (Russian original published in Moscow by Fizmatgiz in 1960).

[15] R.M. Trigub, *Approximation of functions by polynomials with integral coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 261-280 (Russian).

CHAPTER II.

APPROXIMATION OF FUNCTIONS BY POLYNOMIALS IN $C[-1,1]$

2.1. Introduction

Polynomial approximation of functions in $C[-1,1]$ has been investigated extensively. The direct result was proved by Timan, Dzjadyk, Freud and Brudnyi, (see [5] and [7]), each extending the previous work. They proved that for every function $f \in C[-1,1]$ there exists a sequence of polynomials of degree n, P_n, satisfying

$$\begin{aligned} |f(x) - P_n(x)| &\leq C(r)\omega^r\left(f, n^{-1}(n^{-1} + \sqrt{1-x^2})\right) \\ &\equiv C(r)\omega^r(f, n^{-1}\delta_n(x)) \end{aligned} \tag{2.1.1}$$

where

$$\omega^r(f,t) = \sup_{0<h\leq t} \|\Delta_h^r f\|_{C[-1,1]} \tag{2.1.2}$$

and

$$(2.1.3)\qquad \Delta_h^r f(x) = \begin{cases} \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\big(x + (k - \frac{r}{2})h\big) & \text{if} \quad |x \pm \frac{rh}{2}| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recently, it was shown [4, Chapter 7] that there exists a sequence of polynomials of degree n (the best polynomial approximants to f in $C[-1,1]$ would do) that satisfy

$$(2.1.4)\qquad \|f - P_n\|_{C[-1,1]} \le C(r)\omega_\varphi^r(f, n^{-1})$$

where

$$(2.1.5)\qquad \omega_\varphi^r(f,t) = \sup_{|h|\le t} \big(\|\Delta_{h\varphi}^r f\|_{C[-1,1]};\ \varphi(x) = \sqrt{1-x^2}\big).$$

We will show that for every $f \in C[-1,1]$ and $0 \le \lambda \le 1$ there exists a sequence of polynomials of degree n satisfying

$$(2.1.6)\qquad |f(x) - P_n(x)| \le C(r,\lambda)\omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda})$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$ and

$$(2.1.7)\qquad \omega_{\varphi^\lambda}^r(f,t) = \sup_{|h|\le t} \big(\|\Delta_{h\varphi^\lambda}^r f\|_{C[-1,1]};\ \varphi(x) = \sqrt{1-x^2}\big).$$

The inequality (2.1.6) actually coincides with (2.1.1) when $\lambda = 0$ and with (2.1.4) when $\lambda = 1$.

Converse inequalities for both (2.1.1) and (2.1.4) of the weak-type variety were proved. We will show that the existence of a sequence of polynomials $P_n(x)$ satisfying

$$|f(x) - P_n(x)| \leq C\Psi\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \tag{2.1.8}$$

with $\Psi(t)$ an increasing function satisfying

$$\Psi(\mu t) \leq C(\mu^r + 1)\Psi(t) \qquad \mu, t > 0 \tag{2.1.9}$$

where C is independent of μ and t, implies

$$\omega^r_{\varphi^\lambda}(f,t) \leq C(r,\lambda)t^r \sum_{0<n\leq t^{-1}} (n+1)^{r-1}\Psi(n^{-1}). \tag{2.1.10}$$

This result includes results of Timan and others for $\lambda = 0$ and a result of Totik and the first author for $\lambda = 1$. One has to emphasize that while results by Ditzian and Totik in [4] apply to the case $\lambda = 1$ for $1 \leq p \leq \infty$, we have no hope of proving a general result (i.e. for $p \neq \infty$ and $0 \leq \lambda < 1$), as in the case $\lambda = 0$ such a result was proved to be impossible (see [6] and [1]).

Results about the approximation of the derivatives and an estimate of the derivatives of the approximating polynomials will be given in section 2.6.

2.2. The direct estimate

The direct result already stated in the introduction as (2.1.6) will be stated and proved here utilizing a crucial lemma which will be proved in the next section.

Theorem 2.2.1. *For* $0 \leq \lambda \leq 1$, *an integer* r, *and any* $f \in C[-1,1]$ *we have a sequence of polynomials* $P_n(x)$ *of degree* n *satisfying*

$$|f(x) - P_n(x)| \leq C(r,\lambda)\omega^r_{\varphi^\lambda}\left(f, n^{-1}\delta_n(x)^{1-\lambda}\right), \tag{2.2.1}$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$.

For the proof we will need the following lemma.

Lemma 2.2.2. *For* $0 \leq \lambda \leq 1$, *an integer* r *and a function* $f \in C[-1,1]$ *we can construct a function* $G_n(x)$ *such that*

$$|f(x) - G_n(x)| \leq M\omega^r_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}) \tag{2.2.2}$$

and

$$\delta_n(x)^r |G_n^{(r)}(x)| \leq Mn^r\omega^r_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}), \tag{2.2.3}$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$.

Proof of Theorem 2.2.1: Using Lemma 2.2.2, we approximate $f - G_n$ by the polynomial 0. We now follow [4, Chapter 7] to approximate G_n. We first approximate $G_n^{(r-1)}$ by $P_{n,r-1}$. We need a trigonometric polynomial kernel T_n

which satisfies

$$T_n(x) \geq 0, \qquad \int_{-\pi}^{\pi} T_n(x)\,dx = 1$$

(2.2.4) and

$$\int_{-\pi}^{\pi} |x|^{2r+3} T_n(x)\,dx \leq Ln^{-2r-3}.$$

(Such polynomials are constructed in [5, p.57]). We now define $P_{n,r-1}$ by

$$P_{n,r-1}(x) \equiv \int_{-\pi}^{\pi} G_n^{(r-1)}\big(\cos((\arccos x) - t)\big) T_n(t)\,dt.$$

Denoting

(2.2.5) $$\Delta_{\frac{1}{t}}(x) = |t|(|t| + \sqrt{1-x^2}) \qquad \text{and} \qquad \delta_n(x) = n\Delta_n(x),$$

we now have

$$\begin{aligned} I_n(x) &\equiv \left|\delta_n(x)^{r-1}(G_n^{(r-1)}(x) - P_{n,r-1}(x))\right| \\ &= \left|\delta_n(x)^{r-1} \int_{-\pi}^{\pi} \left[\int_{\cos((\arccos x)-t)}^{x} G_n^{(r)}(u)\,du\right] T_n(t)\,dt\right|. \end{aligned}$$

Using straightforward computation, we have

$$\Delta_{\frac{1}{t}}(x) \geq |x(1-\cos t) \mp \sqrt{1-x^2}\sin t| = |x - \cos\big((\arccos x) \pm t\big)|$$

(see [4, p.80]), and hence

$$I_n(x) \le \int_{-\pi}^{\pi} \frac{1}{|x - \cos((\arccos x) - t)|} \times \left| \int_{\cos((\arccos x)-t)}^{x} \delta_n(x)^{r-1} \Delta_{\frac{1}{t}}(x) G_n^{(r)}(u)\, du \right| T_n(t)\, dt.$$

We recall (7.2.5) of [4], that is,

$$\delta_n(x) \le 9 \max(n^2t^2, 1)\delta_n(u)$$

and

$$\Delta_{\frac{1}{t}}(x) \le 10|t| \max(n^2t^2, 1)\delta_n(u)$$

for u between x and $\cos((\arccos x) - t)$, and the elementary estimate

$$\Delta_{\frac{1}{t}}(x) \le |t|(|t| + \sqrt{1 - x^2}) \le |t|(n|t| + 1)\delta_n(x).$$

These estimates now imply

$$I_n(x) \le \int_{-\pi}^{\pi} \frac{1}{|x - \cos((\arccos x) - t)|} \left| \int_{\cos(\arccos x)-t)}^{x} |t|(n|t| + 1) \times \left(\frac{\delta_n(x)}{\delta_n(u)}\right)^r |\delta_n(u)^r G_n^{(r)}(u)|\, du \right| T_n(t)\, dt.$$

Using Lemma 2.2.2 and

(2.2.6) $$\omega^r_{\varphi^\lambda}(f, t) \sim K_{r,\varphi^\lambda}(f, t^r)$$

where

$$K_{r,\varphi^\lambda}(f,t^r) = \inf_{g\in C^r}(\|f-g\| + t^r\|\varphi^{\lambda r}g^{(r)}\|), \tag{2.2.7}$$

we have

$$\begin{aligned}
|\delta_n^r(u)G_n^{(r)}(u)| &\le Mn^r\omega_{\varphi^\lambda}^r(f,n^{-1}\delta_n(u)^{1-\lambda}) \\
&\le M_1n^rK_{r,\varphi^\lambda}(f,n^{-r}\delta_n(u)^{r(1-\lambda)}) \\
&\le M_1n^rK_{r,\varphi^\lambda}\Big(f,n^{-r}\Big(\frac{\delta_n(u)}{\delta_n(x)}\Big)^{r(1-\lambda)}\delta_n(x)^{r(1-\lambda)}\Big) \\
&\le M_1n^r\Big(\Big(\frac{\delta_n(u)}{\delta_n(x)}\Big)^{r(1-\lambda)}+1\Big)K_{r,\varphi^\lambda}(f,n^{-r}\delta_n(x)^{r(1-\lambda)}) \\
&\le M_2n^r\Big(\Big(\frac{\delta_n(u)}{\delta_n(x)}\Big)^{r(1-\lambda)}+1\Big)\omega_{\varphi^\lambda}^r(f,n^{-1}\delta_n(x)^{1-\lambda}).
\end{aligned}$$

We can now write

$$\begin{aligned}
I_n(x) &\le M_2n^r\omega_{\varphi^\lambda}^r(f,n^{-1}\delta_n(x)^{1-\lambda})\int_{-\pi}^{\pi}\frac{1}{|x-\cos((\arccos x)-t)|} \\
&\qquad\times\Big|\int_{\cos((\arccos x)-t)}^{x}|t|(n|t|+1)\big((\frac{\delta_n(x)}{\delta_n(u)})^{\lambda r}+(\frac{\delta_n(x)}{\delta_n(u)})^{r}\big)\,du\Big|\,T_n(t)\,dt \\
&\le M_3n^r\omega_{\varphi^\lambda}^r(f,n^{-1}\delta_n(x)^{1-\lambda})\int_{-\pi}^{\pi}|t|(n|t|+1)(n^{2r}t^{2r}+1)T_n(t)\,dt \\
&\le M_4n^{r-1}\omega_{\varphi^\lambda}^r(f,n^{-1}\delta_n(x)^{1-\lambda}).
\end{aligned}$$

We have in fact completed the proof of our theorem for the case $r=1$. For $r>1$ we continue the process dealing with $\delta_n(x)^{r-1}(G_n^{(r-1)}(x)-P_{n,r-1}(x))$ instead of

$\delta_n(x)^r G_n^{(r)}(x)$ and obtaining the estimate for $G_n^{(r-2)}(x)$ using

$$G_{n,r-1}(x) \equiv \int_0^x (G_n^{(r-1)}(u) - P_{n,r-1}(u))\,du$$

$$\equiv G_n^{(r-2)}(x) - \int_0^x P_{n,r-1}(u)\,du + C_n$$

and

$$|\delta_n(x)^{r-2}(G_{n,r-1}(x) - P_{n,r-2}(x))| \leq Mn^{r-2}\omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda}).$$

Repeating the process, we have

$$|G_n(x) - P_{n+r-1}(x)| \leq C\omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda})$$

$$\leq C_1\omega_{\varphi^\lambda}^r(f, (n+r-1)^{-1}\delta_{n+r-1}(x)^{1-\lambda}).$$

As the result is valid for all n, we have proved our theorem pending the proof of Lemma 2.2.2. □

2.3. Construction of the auxiliary function G_n

In this section we will prove Lemma 2.2.2 using the technique employed in [2] and [4]. For the sake of completeness and as the situation here is simpler and somewhat different we will give the complete proof.

Proof of Lemma 2.2.2: First we choose $\ell \equiv \ell(n)$ such that $4^{\ell-1} < n^2 \leq 4^\ell$,

and then we write

$$
\begin{aligned}
G_n(x) = \Psi_0(x) f_{\frac{1}{n}}(x) + \sum_{k=-\ell+1}^{\ell-1} \Psi_k(x)(1 - \Psi_{k+\operatorname{sgn} k}(x)) f_{\tau_k}(x) \\
+ \Psi_\ell(x) f_{\tau_\ell}(x) + \Psi_{-\ell}(x) f_{\tau_{-\ell}}(x)
\end{aligned} \tag{2.3.1}
$$

where

$$
\begin{aligned}
f_\tau(x) = r^r \int_0^{1/r} \cdots \int_0^{1/r} \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} \\
\times f(x + j\tau(u_1 + \cdots + u_r))\, du_1 \ldots du_r,
\end{aligned} \tag{2.3.2}
$$

$$
\tau_k \equiv \tau_k(n) = n^{-1} 2^{-|k|} \operatorname{sgn} k,
$$

$\Psi(x) \in C^\infty(R)$, $\Psi(x) = 1$ for $x \le 1$, $0 \le \Psi(x) \le 1$ for $1 < x < 3$ and $\Psi(x) = 0$ for $x > 3$ and

$$
\Psi_k(x) = \begin{cases} \Psi(4^{-k}(1-x)) & k \le -1 \\ \Psi(4^k(1+x)) & k \ge 1 \\ 1 - \Psi_1(x) - \Psi_{-1}(x) & k = 0\,. \end{cases}
$$

We will only estimate (2.2.2) and (2.2.3) for $x \in I_k \equiv [-1 + 4^{-k-1}, -1 + 4^{-k}]$ with $k = 1, \ldots, \ell - 1$ and for $I_\ell^* \equiv [-1, -1 + 4^{-\ell}]$ as the estimates for $[3/4, 1]$ are symmetric to that in $[-1, -3/4]$ and since the estimates in $[-3/4, 3/4]$ are very simple. (We note that the first term of (2.3.1), i.e. $\Psi_0(x) f_{\frac{1}{n}}(x)$ spoils the symmetry of the expression $G_n(x)$ but this will not negate the above as $\operatorname{supp} \Psi_0(x) \subset [-3/4, 3/4]$.) We further observe that $\operatorname{supp} \Psi_k(x)(1 - \Psi_{k+1}(x)) \subset [-1+4^{-k-1}, -1+3\cdot 4^{-k}]$ and $(\operatorname{supp} \Psi_\ell(x)) \cap [-1, 1] = [-1, -1+3\cdot 4^{-\ell}]$. Therefore,

we can write for $x \in I_k = [-1+4^{-k-1}, -1+4^{-k}]$ and $1 \le k \le \ell - 1$,

$$|f(x) - G_n(x)| \le \max_{j=k,k+1} |f(x) - f_{\tau_j}(x)| \le \sup_{0<h<\tau_k} |\vec{\Delta}_h^r f(x)| \tag{2.3.3}$$

where

$$\vec{\Delta}_h f(x) \equiv f(x+h) - f(x) \qquad \text{and} \qquad \vec{\Delta}_h^r f(x) \equiv \vec{\Delta}_h(\vec{\Delta}_h^{r-1} f(x)).$$

Since for $x \in I_k$,

$$\sup_{0<h\le\tau_k} h/\varphi(x+\frac{r}{2}h)^\lambda \le \sup_{0<h\le\tau_k} h/2^{-(k+1)\lambda} \le 2^\lambda n^{-1} 2^{-k(1-\lambda)}$$

we have

$$\begin{aligned} |f(x) - G_n(x)| &\le \sup_{0<h\le\tau_k} |\Delta_h^r f(x+\frac{r}{2}h)| \\ &\le \sup_{\xi} \sup_{0<\eta\le 2^\lambda n^{-1}2^{-k(1-\lambda)}} |\Delta^r_{\eta\varphi(\xi)^\lambda} f(\xi)| \\ &\le \sup_{0<\eta\le 2n^{-1}\delta_n(x)^{1-\lambda}} \|\Delta^r_{\eta\varphi^\lambda} f\| \\ &= \omega^r_{\varphi^\lambda}(f, 2n^{-1}\delta_n(x)^{1-\lambda}). \end{aligned} \tag{2.3.4}$$

For $x \in I_\ell^* = [-1, -1+4^{-\ell}]$, we note that when $h \le \dfrac{2}{r}$

$$h/\varphi(x+\frac{r}{2}h)^\lambda \le h/\varphi(-1+\frac{r}{2}h)^\lambda \le h/(1-(-1+\frac{r}{2}h)^2)^{\lambda/2}$$

$$\le h/(\frac{r}{2}h)^{\lambda/2} = \left(\frac{2}{r}\right)^{\lambda/2} h^{1-(\lambda/2)}$$

and as $0 < h \le \tau_\ell$ the condition $h \le 2/r$ means that $n^{-2} < 4/r$. We can now

write for $x \in I_\ell^*$

$$
\begin{aligned}
|f(x) - G_n(x)| &\le \sup_{0<h\le\tau_\ell} |\vec{\Delta}_h^r f(x)| = \sup_{0<h\le\tau_\ell} |\Delta_h^r f(x + \frac{r}{2}h)| \\
&\le \sup_{\xi} \sup_{0<\eta\le(\frac{2}{r})^\lambda \tau_\ell^{1-\lambda/2}} |\Delta_{\eta\varphi(\xi)^\lambda}^r f(\xi)| \\
&= \omega_{\varphi^\lambda}^r(f, (\frac{2}{r})^\lambda n^{-1}\delta_n(x)^{1-\lambda}) \\
&\le \omega_{\varphi^\lambda}^r(f, 2n^{-1}\delta_n(x)^{1-\lambda}).
\end{aligned} \tag{2.3.5}
$$

We now use Thorem 4.1.2 of [4] with the step weight φ^λ (which satisfies the conditions on step weights there) to obtain

$$
\omega_{\varphi^\lambda}^r(f, 2h) \le C\omega_{\varphi^\lambda}^r(f, h) \tag{2.3.6}
$$

where C is independent of f and h. This implies (2.2.2) for $n \ge n_0$. To obtain (2.2.2) for all n we increase the constant.

For the estimate (2.2.3) we first recall that for $\tau > 0$

$$
|f_\tau^{(r)}(x)| \le \sum_{j=1}^{r} \binom{r}{j} \left(\frac{r}{j}\right)^r \tau^{-r} |\vec{\Delta}_{j\tau/r}^r f(x)|. \tag{2.3.7}
$$

Using (2.3.7) and earlier considerations (see (2.3.5)) we obtain for $x \in I_\ell^*$

$$
\begin{aligned}
|G_n^{(r)}(x)| = |f_{\tau_\ell}^{(r)}(x)| &\le \tau_\ell^{-r} \Big\{ \sum_{j=1}^{r} \binom{r}{j} r^r \Big\} \sup_{0<h\le\tau_\ell} |\vec{\Delta}_h^r f(x)| \\
&\le \tau_\ell^{-r} (2r)^r \omega_{\varphi^\lambda}^r(f, 2n^{-1}\delta_n(x)^{1-\lambda}).
\end{aligned}
$$

Using (2.3.6) and $\tau_\ell^{-1} \le 2n^2 \le 6n\delta_n(x)^{-1}$ for $x \in I_\ell^*$ we have

$$|G_n^{(r)}(x)| \le C_1 r^r n^r \delta_n(x)^{-r} \omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda}).$$

To prove (2.2.3) for $x \in I_k$ $1 \le k \le \ell - 1$, we write

$$G_n(x) = f_{\tau_k}(x) + \Psi_{k+1}(x)(f_{\tau_{k+1}}(x) - f_{\tau_k}(x)). \tag{2.3.8}$$

For $x \in I_k$ we show using the technique employed in proving (2.3.4)

$$|f_{\tau_k}^{(r)}(x)| \le \tau_k^{-r}(2r)^r \sup_{0<h\le\tau_k} |\vec{\Delta}_h^r f| \le M(n\delta_n(x)^{-1})^r \omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda})$$

and

$$|f_{\tau_{k+1}}^{(r)}(x)| \le M(n\delta_n(x)^{-1})^r \omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda})$$

with M independent of k and f.

Since for $x \in I_k$

$$|\Psi_{k+1}^{(m)}(x)| \le M_1 4^{km} \le M_2(n\delta_n(x)^{-1})^m$$

we have to show only

$$|f_{\tau_k}^{(s)}(x) - f_{\tau_{k+1}}^{(s)}(x)| \le M_3(n\delta_n(x)^{-1})^s \omega_{\varphi^\lambda}^r(f, n^{-1}\delta_n(x)^{1-\lambda}) \tag{2.3.9}$$

for $0 \le s \le r$. For $s = 0$ we derive this estimate following (2.3.4) and using

$$|f_{\tau_k}(x) - f_{\tau_{k+1}}(x)| \le |f_{\tau_k}(x) - f(x)| + |f_{\tau_{k+1}}(x) - f(x)|.$$

For $s = r$ we already obtained the estimate as we may write

$$|f^{(r)}_{\tau_k}(x) - f^{(r)}_{\tau_{k+1}}(x)| \le |f^{(r)}_{\tau_k}(x)| + |f^{(r)}_{\tau_{k+1}}(x)|.$$

For other s (2.3.9) follows using the estimate

$$\|g^{(s)}\|_{C[a,b]} \le A[(b-a)^{-s}\|g\|_{C[a,b]} + (b-a)^{r-s}\|g^{(r)}\|_{C[a,b]}]$$

(see Lemma 2.2.1 of [2]) where A is independent of $[a,b]$ and g, $a = -1 + 4^{-k-1}$, $b = -1 + 4^{-k}$ and

$$g(x) = f_{\tau_k}(x) - f_{\tau_{k+1}}(x).$$

□

2.4. Estimate on derivatives of polynomials

In the proof of the converse results we need an estimate on the derivatives of polynomials.

Theorem 2.4.1. *Suppose for a polynomial of degree n we have the estimate*

$$|P_n(x)| \le M\big(n^{-1}\delta_n(x)\big)^{\beta}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \qquad |x| < 1 \tag{2.4.1}$$

where β is a real number and ω satisfies

$$\omega(\mu t) \le C(\mu^s + 1)\omega(t), \qquad \mu > 0, \ \ t > 0. \tag{2.4.2}$$

Then for $\ell \geq \beta + s(1-\lambda)$

$$|P_n^{(\ell)}(x)| \leq M_1 \big(n^{-1}\delta_n(x)\big)^{\beta-\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \qquad |x|<1 \tag{2.4.3}$$

where M_1 *depends on* M, C, ℓ, s, β *and* λ *but not on* x, P_n *or* n.

Proof: Let us begin by estimating $|P_n'(x)|$. It follows from [5, *p.*70 (2)] that for a trigonometric polynomial $T_n(x)$ we have

$$T_n'(t) = 2^{2m}n^{-2m+2}\int_{-\pi}^{\pi} T_n(t+u)H_{n,m}(u)\widetilde{K}_{n,m}(u)\,du$$

where $\widetilde{K}_{n,m}$ and $H_{n,m}$ are trigonometric polynomials,

$$\widetilde{K}_{n,m}(u) = \Big(\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\Big)^{2m} \qquad \text{and} \qquad |H_{n,m}(u)| \leq 1.$$

We recall [5, *p.* 57] that

$$\int \widetilde{K}_{n,m}(u)\,du \leq c(m)n^{2m-1},$$

and hence we have

$$|T_n'(t)| \leq C_1(m)n\int_{-\pi}^{\pi} |T_n(t+u)|S_{nm}(u)\,du$$

with $S_{nm}(u)$ an even positive trigonometric polynomial of degree $(n-1)m$ sat-

isfying

$$\int_{-\pi}^{\pi} S_{n,m}(u)\,du = 1 \qquad \text{and} \tag{2.4.4}$$

$$\int_{-\pi}^{\pi} |u|^{\gamma} S_{n,m}(u)\,du \sim (\frac{1}{n})^{\gamma}, \quad 0 < \gamma < 2m-1.$$

We set $T_n(t) = P_n(\cos t)$, and hence

$$\begin{aligned} |T_n'(t)| &= |P_n'(\cos t)\sin t| \\ &\le C_1(m)n \int_{-\pi}^{\pi} |P_n(\cos(t+u))| S_{nm}(u)\,du \qquad (2.4.5) \\ &\le MC_1(m)n \int_{-\pi}^{\pi} \big(\frac{1}{n}(\frac{1}{n} + |\sin(t+u)|)\big)^{\beta} \\ &\qquad \times \omega\big(n^{-1}(n^{-1} + |\sin(t+u)|)^{1-\lambda}\big) S_{nm}(u)\,du. \end{aligned}$$

We have

$$\frac{\frac{1}{n} + |\sin(t+u)|}{\frac{1}{n} + |\sin t|} \le \frac{\frac{1}{n} + |\sin t| + |\sin u|}{\frac{1}{n} + |\sin t|} \le 1 + n|\sin u| \le 1 + n|u|,$$

and setting $\tau = t + u$ or calculating directly,

$$\frac{\frac{1}{n} + |\sin t|}{\frac{1}{n} + |\sin(t+u)|} \le 1 + n|u|\,.$$

Therefore, we have

$$\begin{aligned} \omega\big(n^{-1}(n^{-1}+|\sin(t+u)|)^{1-\lambda}\big) &\le \omega\big(n^{-1}(n^{-1} + |\sin t|)^{1-\lambda}(1+n|u|)^{1-\lambda}\big) \\ &\le C\big(1 + (1+n|u|)^{(1-\lambda)s}\big)\omega\big(n^{-1}(n^{-1} + |\sin t|)^{1-\lambda}\big) \end{aligned}$$

and

$$\left(n^{-1}(n^{-1}+|\sin(t+u)|)\right)^{\beta} \le (1+n|u|)^{|\beta|}\left(n^{-1}(n^{-1}+|\sin t|)\right)^{\beta}.$$

Using the above estimates in (2.4.5) and $|\sin t| = \sqrt{1-x^2}$, we have for $2m-1 > |\beta| + |s|(1-\lambda)$

$$\begin{aligned}|P_n'(x)|\sqrt{1-x^2} &\le M_1 n\left(n^{-1}\delta_n(x)\right)^{\beta}\omega\left(n^{-1}\delta_n(x)^{1-\lambda}\right) \\ &\qquad \times \int_{-\pi}^{\pi} S_{nm}(u)(1+n|u|)^{|\beta|}(1+(1+n|u|)^{s})^{1-\lambda}\,du \\ &\le M_2 n\left(n^{-1}\delta_n(x)\right)^{\beta}\omega\left(n^{-1}\delta_n(x)^{1-\lambda}\right).\end{aligned}$$

We now observe that for $1-x^2 \ge \dfrac{1}{n^2}$ we have $\sqrt{1-x^2} \ge \dfrac{1}{2}\left(n^{-1}+\sqrt{1-x^2}\right)$, and hence

$$|P_n'(x)| \le M_3\left(n^{-1}\delta_n(x)\right)^{\beta-1}\omega\left(n^{-1}\delta_n(x)^{1-\lambda}\right) \qquad \text{for} \quad 1-x^2 \ge \frac{1}{n^2}.$$

We set a so that $1-\frac{1}{n^2} = a$ and obtain for $|x| \le a$

$$|P_n'(x)| \le M_4(n^{-1}(n^{-1}+\sqrt{a^2-x^2}))^{\beta-1}\omega(n^{-1}(n^{-1}+\sqrt{a^2-x^2})^{1-\lambda})$$

or for $|x| < 1$

$$|P_n'(ax)| \le M_5\left(n^{-1}\delta_n(x)\right)^{\beta-1}\omega\left(n^{-1}\delta_n(x)^{1-\lambda}\right), \tag{2.4.6}$$

and therefore, iterating (2.4.6), we have

$$|P_n^{(\ell)}(a^{\ell}x)| \le M_6\left(n^{-1}\delta_n(x)\right)^{\beta-\ell}\omega\left(n^{-1}\delta_n(x)^{1-\lambda}\right), \qquad |x| < 1. \tag{2.4.7}$$

Hence for $|x| \le a^\ell$

$$|P_n^{(\ell)}(x)| \le M_7\big(n^{-1}\delta_n(x)\big)^{\beta-\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big). \tag{2.4.8}$$

Condition (2.4.2) now implies for $0 < t < t_1$

$$\frac{\omega(t_1)}{(t_1)^s} \le 2C\frac{\omega(t)}{t^s} \tag{2.4.9}$$

(with C of (2.4.2)), and therefore, setting

$$t_1 = n^{-1}\delta_n(x)^{1-\lambda} \qquad \text{for} \quad |x| \le a^\ell$$

and $t = n^{-2+\lambda}$, and recalling $\beta - \ell + (1-\lambda)s \le 0$, we have

$$\begin{aligned} |P_n^{(\ell)}(x)| &\le M_8 n^{\ell+(1-\lambda)s-\beta}\delta_n(x)^{\beta-\ell+(1-\lambda)s}\omega(n^{-2+\lambda}) \\ &\le M_9 n^{2(\ell-\beta)}\omega(n^{-2+\lambda}), \qquad |x| < a^\ell. \end{aligned}$$

Using Lemma 8.4.3 of [4],

$$|P_n^{(\ell)}(x)| \le M_{10}n^{2(\ell-\beta)}\omega(n^{-2+\lambda}) \qquad |x| \le 1,$$

which, combined with (2.4.8), implies (2.4.3). □

2.5. The converse result

In this section we will prove a converse result that will unify the result of [4] for the particular case $p = \infty$ with the classical converse result for polynomial

approximation (see [5] and [7]). Of course for $1 \leq p < \infty$ only the result in [4] stands.

Theorem 2.5.1. *Suppose for* $f \in C[-1,1]$ *and some given* λ, $0 \leq \lambda \leq 1$ *there exists a sequence of polynomials satisfying*

$$|f(x) - P_n(x)| \leq M\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \tag{2.5.1}$$

where $\omega(t)$ *is an increasing function satisfying (2.4.2) with* $s = r$, *that is,*

$$\omega(\mu t) \leq C(\mu^r + 1)\omega(t)$$

for any $\mu > 0$, $t > 0$, *with* C *independent of* μ *and* t. *Then for any* $t > 0$

$$\omega^r_{\varphi^\lambda}(f,t) \leq Mt^r \sum_{0<n\leq t^{-1}} (n+1)^{r-1}\omega(n^{-1}). \tag{2.5.2}$$

We will also need the following lemma from [5, *p.*58].

Lemma 2.5.2. *Suppose* $0 < u_k \leq u_{k+1} \leq \cdots \leq u_\ell$ *such that* $2 \leq u_k/u_{k-1} \leq 4$ *and suppose that* $\omega(u)$ *is an increasing function. Then*

$$\sum_{k=i}^{\ell} u_k^r \omega(u_k^{-1}) \leq M \sum_{[\frac{1}{2}u_i]\leq n\leq u_\ell} (n+1)^{r-1}\omega(n^{-1}). \tag{2.5.3}$$

Proof of Theorem 2.5.1: The proof of our theorem, i.e., of (2.5.2), starts in a way quite similar to that of the cases $\lambda = 0$ [5, *p.*73] and $\lambda = 1$ [4, *p.*83]. We

write for x and h satisfying

$$I(x,h,\lambda,r) \equiv [x - \frac{r}{2}h\varphi^{\lambda}, x + \frac{r}{2}h\varphi^{\lambda}] \subset [-1,1] \tag{2.5.4}$$

$$\begin{aligned} |\Delta^r_{h\varphi^\lambda} f(x)| &\le |\Delta^r_{h\varphi^\lambda}(f - P_{2^\ell})(x)| + |\Delta^r_{h\varphi^\lambda} P_{2^\ell}(x)| \\ &\le 2^r \max_{\xi \in I(x,h,\lambda,r)} |(f - P_{2^\ell})(\xi)| \\ &\quad + h^r \varphi(x)^{\lambda r} \max_{\xi \in I(x,h,\lambda,r)} |P^{(r)}_{2^\ell}(\xi)|, \end{aligned} \tag{2.5.5}$$

and recall that when $I(x,h,\lambda,r) \not\subset [-1,1]$, $\Delta^r_{h\varphi^\lambda} f(x) = 0$. Given x and h for which (2.5.4) is satisfied, we choose an integer ℓ which is maximal satisfying

$$2^{-\ell}\big(2^{-\ell} + \varphi(x)\big)^{1-\lambda} \ge 3rh. \tag{2.5.6}$$

We will show that for such an ℓ and $\xi \in I(x,h,\lambda,r)$ (satisfying (2.5.4)) we have

$$h \le 2^{-\ell}\big(2^{-\ell} + \varphi(\xi)\big)^{1-\lambda} \le 24rh \tag{2.5.7}$$

and

$$\frac{1}{3r}\big(2^{-\ell} + \varphi(x)\big) \le 2^{-\ell} + \varphi(\xi) \le 2\big(2^{-\ell} + \varphi(x)\big). \tag{2.5.8}$$

To prove the inequalities on the left of (2.5.7) we use (2.5.6) and prove the inequality on the left of (2.5.8). The latter is evident for $\varphi(x) \le r2^{-\ell+1}$ as $\varphi(\xi) \ge 0$ and $2r+1 \le 3r$. The inequality $\varphi(x) > r2^{-\ell+1}$ implies $rh\varphi(x)^{\lambda} \le \frac{1}{2}\varphi(x)^2$. To show

this we write, using (2.5.6),

$$2rh \le \frac{2}{3}2^{-\ell}\big(2^{-\ell}+\varphi(x)\big)^{1-\lambda} \le \frac{2}{3}\frac{1}{2r}\varphi(x)\big(\frac{1}{2r}+1\big)^{1-\lambda}\varphi(x)^{1-\lambda} \le \frac{1}{2}\varphi(x)^{2-\lambda}.$$

The identity $\varphi(\xi)^2 = \varphi(x)^2 + (x-\xi)(\xi+x)$ implies

$$\varphi(\xi)^2 \ge \varphi(x)^2 - rh\varphi^\lambda(x) \ge \varphi(x)^2 - \frac{1}{2}\varphi(x)^2 = \frac{1}{2}\varphi(x)^2,$$

which, using $\sqrt{2} \le 3r$, completes the proof of the left side of (2.5.8) and therefore of (2.5.7). To prove the inequality on the right of (2.5.7), we observe that the choice of ℓ as maximal satisfying (2.5.6) implies

$$2^{-\ell}\big(2^{-\ell}+\varphi(x)\big)^{1-\lambda} \le 12rh,$$

and we complete the proof of (2.5.7) when we prove the right hand side of (2.5.8) (for $\xi \in I(x,h,\lambda,r) \subset [-1,1]$). If $\varphi(\xi)^2 \le \frac{3}{2}rh\varphi(x)^\lambda$, we recall that

$$\varphi(x)^2 \ge \min(|1-x|\,|1+x|) \ge \frac{r}{2}h\varphi(x)^\lambda \ge \frac{1}{3}\varphi(\xi)^2 \qquad \text{or} \qquad \varphi(\xi) \le \sqrt{3}\varphi(x).$$

If $\varphi(\xi)^2 > \frac{3}{2}rh\varphi(x)^\lambda$, we write

$$\varphi(\xi)^2 \le \varphi(x)^2 + 2|\xi - x| \le \varphi(x)^2 + rh\varphi(x)^\lambda \qquad \text{or} \qquad \varphi(\xi)^2 < 3\varphi(x)^2.$$

We now use (2.4.2), (2.5.7) and the fact that ω is increasing, and obtain

$$\begin{aligned}
&\max_{\xi\in I(x,h,\lambda,r)}|f-P_{2^\ell}(\xi)|\\
&\qquad\le M\max_{\xi\in I(x,h,\lambda,r)}\omega\big(2^{-\ell}\big(2^{-\ell}+\sqrt{1-\xi^2}\big)^{1-\lambda}\big) \qquad (2.5.9)\\
&\qquad\le M\omega(24rh)\le M_1\omega(h).
\end{aligned}$$

To estimate $|P^{(r)}_{2^\ell}(\xi)|$, we write, recalling $P^{(r)}_1(\xi)=0$,

$$P^{(r)}_{2^\ell}(\xi)=\sum_{k=1}^{\ell}\{P^{(r)}_{2^k}(\xi)-P^{(r)}_{2^{k-1}}(\xi)\}.$$

As

$$\begin{aligned}
|P_{2^k}(x)-P_{2^{k-1}}(x)|&\le|P_{2^k}(x)-f(x)|+|P_{2^{k-1}}(x)-f(x)|\\
&\le 2M\omega\big(2^{-k+1}\delta_{2^{-k+1}}(x)^{1-\lambda}\big),
\end{aligned}$$

Theorem 2.4.1 with $\beta=0$, $\ell=s=r$ and $n=2^k$ implies

$$\begin{aligned}
|P^{(r)}_{2^k}(\xi)-P^{(r)}_{2^{k-1}}(\xi)|&\le M_2\big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)\big)^{-r}\omega\big(2^{-k+1}\big(2^{-k+1}+\varphi(\xi)\big)^{1-\lambda}\big)\\
&\le M_2\big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)\big)^{-r}\omega\big(2^{-k+1}\big(2^{-k+1}+\varphi(\xi)\big)^{1-\lambda}\big).
\end{aligned}$$

Therefore, using (2.5.8) and (2.4.2), we have

$$h^r\varphi(x)^{\lambda r}|P_{2^\ell}^{(r)}(\xi)| \le h^r\big(2^{-\ell}+\varphi(x)\big)^{\lambda r}|P_{2^\ell}^{(r)}(\xi)|$$

$$\le M_3 h^r\big(2^{-\ell}+\varphi(\xi)\big)^{\lambda r}$$

$$\times\sum_{k=1}^{\ell}\big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)\big)^{-r}\omega\big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\big)$$

$$\le M_3 h^r\sum_{k=1}^{\ell}\big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\big)^{-r}\omega\big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\big).$$

We now set in Lemma 2.5.2 $u_k^{-1} = [2^{-k}(2^{-k}+\varphi(\xi))^{1-\lambda}]$, and observe that $2 \le u_k/u_{k-1} \le 2\cdot 2^{1-\lambda} \le 4$. We further recall, using (2.5.7), that $u_k^{-1} \le u_\ell^{-1} \le h^{-1}$, and hence we can write

(2.5.10)
$$h^r\varphi(x)^{\lambda}|P_{2^\ell}^{(r)}(\xi)|$$

$$\le M_3 h^r \sum_{[2^{-k}(2^{-k}+\varphi(\xi))^{1-\lambda}]\ge h}\big(2^{-k}\delta_{2^{-k}}(\xi)^{1-\lambda}\big)^r\omega\big(2^{-k}\delta_{2^{-k}}(\xi)^{1-\lambda}\big)$$

$$\le M_4 h^r \sum_{0<n\le h^{-1}}(n+1)^r\omega(n^{-1}).$$

Combining (2.5.9) and (2.5.10), we complete the proof of our theorem. □

2.6. The derivatives of the approximating polynomial

In this section we will estimate $f^{(r)}-P_n^{(r)}$ in terms of the estimate of $f-P_n$ in analogy to a classical result of Timan and others (see [5, *p.*74, Theorem 5]). The

L_p case will also be stated and proved. We will then estimate $P_n^{(r)}$ in terms of the estimate of $|f(x) - P_n(x)|$.

Theorem 2.6.1. *Suppose for some* $f \in C[-1,1]$ *and* $0 \le \lambda \le 1$ *there exists a sequence of polynomials* P_n *satisfying*

$$|f(x) - P_n(x)| \le M\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \qquad |x| \le 1 \tag{2.6.1}$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$, $\omega(t)$ *is an increasing function satisfying (2.4.2), with* $s = r$ *and* $\sum_{k=1}^{\infty} k^{r-1}\omega(k^{-1}) < \infty$. *Then* f *has locally* r *continuous derivatives and*

$$|\varphi(x)^{\lambda r}[f^{(r)}(x) - P_n^{(r)}(x)]| \le M_1 \sum_{k > (\frac{1}{n}(\frac{1}{n}+\varphi(x))^{1-\lambda})^{-1}} k^{r-1}\omega(k^{-1}). \tag{2.6.2}$$

Proof: Following the standard technique, we write

$$f = P_n + \sum_{i=1}^{\infty}(P_{2^i n} - P_{2^{i-1}n}).$$

Furthermore, using Lemma 2.5.2 with $u_i^{-1} = \frac{1}{2^i n}(\frac{1}{2^i n} + \varphi(x))^{1-\lambda}$, and letting ℓ there increase to infinity, the inequality $\sum n^{r-1}\omega(\frac{1}{n}) < \infty$, which implies $\sum n^{-1}\omega(\frac{1}{n}) < \infty$, assures us of the convergence of the series above. Using Theorem 2.4.1 (and (2.4.2)), we have

$$\begin{aligned} &|P_{2^i n}^{(r)}(x) - P_{2^{i-1}n}^{(r)}(x)| \\ &\qquad \le M_2\big(2^{-i}n^{-1}\big(2^{-i}n^{-1} + \varphi(x)\big)\big)^{-r}\omega\big(2^{-i+1}n^{-1}(2^{-i+1}n^{-1} + \varphi(x))^{1-\lambda}\big). \end{aligned}$$

The convergence of $\sum n^{r-1}\omega(\frac{1}{n})$ now implies local uniform (in $(x_0-\delta, x_0+\delta) \subset [-1+\delta, 1-\delta]$) convergence of

$$\sum_{i=1}^{\infty}\left(P_{2^i n}^{(r)}(x) - P_{2^{i-1}n}^{(r)}(x)\right),$$

and therefore, $(f-P_n)^{(r)}$ exists locally. We can now write

$$\begin{aligned}
|\varphi^{\lambda r}(f-P_n)^{(r)}(x)| &\le \varphi^{\lambda r}\sum_{i=1}^{\infty}|(P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)})(x)| \\
&\le \varphi^{\lambda r}\sum_{i=1}^{\infty}\left(2^{-i}n^{-1}\delta_{2^i n}(x)\right)^{-r}\omega\left(2^{-i}n^{-1}\delta_{2^i n}(x)^{1-\lambda}\right) \\
&\le \sum_{i=1}^{\infty}\left(2^{-i}n^{-1}\delta_{2^i n}(x)^{1-\lambda}\right)^{-r}\omega\left(2^{-i}n^{-1}\delta_{2^i n}(x)^{1-\lambda}\right) \\
&\le M \sum_{k>[\frac{1}{n}(\frac{1}{n}+\varphi(x))^{1-\lambda}]^{-1}} k^{r-1}\omega(1/k),
\end{aligned}$$

and this completes the proof of our theorem. □

The analogue for L_p was not stated explicitly in [4] or anywhere we know of, so for the sake of completeness, we are stating it here.

Theorem 2.6.2. *Suppose* $f \in L_p[-1,1]$, $1 \le p \le \infty$,

$$E_n(f)_p \equiv \inf \|f - P_n\|_{L_p[-1,1]} \tag{2.6.3}$$

(P_n *polynomial of degree* n) *and*

$$\sum (n+1)^{r-1} E_n(f)_p < \infty.$$

Then $f^{(r)}$ *exists locally in the* L_p *sense and*

$$\|\varphi^r [f^{(r)} - P_n^{(r)}]\|_{L_p[-1,1]} \leq M \sum_{k \geq n} (k+1)^{r-1} E_k(f)_p. \tag{2.6.4}$$

Proof: We follow exactly the classical proof using the series $\sum_{i=1}^{\infty} (P_{2^i n} - P_{2^{i-1} n})$ where P_k is the best polynomial approximant to f in $L_p[-1,1]$. Local L_p convergence and the estimate (2.6.4) itself are guaranteed by the estimate [4, Theorem 8.4.7], i.e.,

$$\|\varphi^r P_n^{(r)}\|_{L_p[-1,1]} \leq C n^r \|P_n\|_{L_p[-1,1]} \tag{2.6.5}$$

for all n-th degree polynomials, P_n, which implies

$$\begin{aligned} \|\varphi^r \sum_{i=1}^{\infty} (P_{2^i n}^{(r)} - P_{2^{i-1} n}^{(r)})\|_{L_p[-1,1]} &\leq \sum_{i=1}^{\infty} \|\varphi^r (P_{2^i n}^{(r)} - P_{2^{i-1} n}^{(r)})\|_p \\ &\leq 2C \sum_{i=1}^{\infty} (2^i n)^r E_{2^i n}(f)_p \\ &\leq C_1 \sum_{k \geq n} (k+1)^{r-1} E_k(f)_p. \end{aligned}$$

□

Theorem 2.6.3. *Suppose for some* $f \in C[-1,1]$ *and some* $0 \le \lambda \le 1$ *the sequence of polynomials* P_n *of degree smaller than* n *satisfies*

$$|f(x) - P_n(x)| \le M\omega^r_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}) \tag{2.6.6}$$

for the r*-th modulus of smoothness* $\omega^r_{\varphi^\lambda}(f,t)$. *Then*

$$|\varphi(x)^{\lambda r}P_n^{(r)}(x)| \le M_1 n^r \delta_n(x)^{(\lambda-1)r}\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big). \tag{2.6.7}$$

Obviously, using Theorem 2.2.1, there exist P_n satisfying (2.6.6). We stated (2.6.6) to stress that it is those P_n that satisfy (2.6.7).

Proof: Following the proof of Theorem 7.3.1 of [4], we write

$$P_n^{(r)} = P_n^{(r)} - P_1^{(r)} = P_n^{(r)} - P_{2^\ell}^{(r)} + \sum_{k=1}^{\ell}\big(P_{2^k}^{(r)} - P_{2^{k-1}}^{(r)}\big)$$

where $2^\ell \le n < 2^{\ell+1}$. As

$$\omega^r_{\varphi^\lambda}(f, \mu t) \le C(\lambda, r)(\mu^r + 1)\omega^r_{\varphi^\lambda}(f,t)$$

for all f, μ and t, we have, using Theorem 2.4.1,

$$
\begin{aligned}
|\varphi(x)^{\lambda(r+1)}P_n^{(r+1)}(x)| \le M_2 \Big[& n^{r+1}\delta_n(x)^{-(r+1)(1-\lambda)}\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big) \\
& + \sum_{k=1}^{\ell} 2^{k(r+1)}(2^{-k}+\varphi)^{-(r+1)(1-\lambda)} \\
& \qquad \times \omega^r_{\varphi^\lambda}(f, 2^{-k}(2^{-k}+\varphi)^{1-\lambda})\Big] \\
& \le M_3 n^{r+1}\delta_n(x)^{-(r+1)(1-\lambda)}\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).
\end{aligned} \tag{2.6.8}
$$

Recalling [4, Theorem 8.4.8], we observe that it is sufficient to estimate $\varphi^{\lambda r}(\frac{1}{n}+\varphi)^{(1-\lambda)r}P_n^{(r)}(x)$ and $\varphi^\alpha P_n^{(r)}(x)$ only for $x \in [-1+4r^2n^{-2}, 1-4r^2n^{-2}]$. We now write

$$
\begin{aligned}
&\Big|n^{-r}\varphi^{\lambda r}\big(n^{-1}+\varphi\big)^{(1-\lambda)r}P_n^{(r)}(x)\Big| \\
&\qquad \le \Big|n^{-r}\varphi^{\lambda r}\big(n^{-1}+\varphi\big)^{(1-\lambda)r}P_n^{(r)}(x) - \Delta^r_{\varphi^\lambda(\frac{1}{n}+\varphi)^{1-\lambda}\frac{1}{n}}P_n(x)\Big| \\
&\qquad\qquad + \big|\Delta^r_{\varphi^\lambda(\frac{1}{n}+\varphi)^{1-\lambda}\frac{1}{n}}(P_n - f)(x)\big| + \big|\Delta^r_{\varphi^\lambda(\frac{1}{n}+\varphi)^{1-\lambda}\frac{1}{n}}f(x)\big| \\
&\qquad \equiv I_1 + I_2 + I_3.
\end{aligned}
$$

Clearly

$$
I_3 \le \omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).
$$

Moreover, for $\xi_j = x + (\frac{r}{2}-j)\varphi(x)^\lambda(\frac{1}{n}+\varphi(x))^{1-\lambda}\frac{1}{n}$ and $x \in [-1+4r^2n^{-2}, 1-4r^2n^{-2}]$ we have

$$
A^{-1}\varphi(x) \le \varphi(\xi_j) \le A\varphi(x), \qquad 0 \le j \le r,
$$

and hence, using $\omega^r_{\varphi^\lambda}(f,\mu t) \le C(\mu^r+1)\omega^r_{\varphi^\lambda}(f,t)$, we have

$$I_2 \le 2^r \max_{0\le j\le r} |P_n(\xi_j)-f(\xi_j)| \le M\omega^r_{\varphi^\lambda}\left(f, n^{-1}\delta_n(\xi_j)^{1-\lambda}\right)$$

$$\le M_3\omega^r_{\varphi^\lambda}\left(f, n^{-1}\delta_n(x)^{1-\lambda}\right).$$

Following Theorem 7.3.1 of [4], we estimate

$$I_1 \le n^{-r}\varphi^{\lambda r}(x)\big(n^{-1}+\varphi(x)\big)^{(1-\lambda)r}|P_n^{(r)}(x)-P_n^{(r)}(\xi)|$$

$$\le n^{-r}\varphi^{\lambda r}(x)\big(n^{-1}+\varphi(x)\big)^{(1-\lambda)r}|x-\xi|\big|P_n^{(r+1)}(\eta)\big|$$

where $\eta,\ \xi \in (x-\frac{r}{2}\varphi^\lambda(x)(\frac{1}{n}+\varphi(x))^{1-\lambda}\frac{1}{n}, x+\frac{r}{2}\varphi^\lambda(x)(\frac{1}{n}+\varphi(x))^{1-\lambda}\frac{1}{n})$. For $x \in [-1+4r^2n^{-2}, 1-4r^2n^{-2}]$, the above estimate and (2.6.8) yield the desired result.

□

2.7. Remarks

Remark 2.7.1. *In Theorems 2.2.1, 2.4.1, 2.5.1 and 2.6.1 we needed a condition on the function majorizing* $|f(x)-P_n(x)|$ *while for the analogues on* $L_p[-1,1]$, $1 \le p \le \infty$ *no condition was imposed on* $E_n(f)$. *The technical reason is that in Theorem 8.4.7 of [4] such a condition is not needed and in our Theorem 2.4.1 it is.*

For the behaviour $\omega(t)=t^\alpha$ we may obtain the following corollaries.

Corollary 2.7.2. *For* $f \in C[-1,1]$, $\alpha < r$ *and* $0 \le \lambda \le 1$ *the conditions*

$$|f(x) - P_n(x)| \le C\big(n^{-1}\big(n^{-1} + \sqrt{1-x^2}\big)^{1-\lambda}\big)^{\alpha}$$
$$\equiv C\big(n^{-1}\delta_n(x)^{1-\lambda}\big)^{\alpha} \tag{2.7.1}$$

and

$$\omega^r_{\varphi^\lambda}(f,t) \le C_1 t^{\alpha} \tag{2.7.2}$$

are equivalent.

Corollary 2.7.3. *For* $f \in C[-1,1]$, $r < \alpha$ *and* $0 \le \lambda \le 1$ *(2.7.1) implies the existence of* $f^{(r)}$ *locally,* $f^{(r-1)} \in A.C_{loc}$ *and*

$$(1-x^2)^{\lambda r/2}|f^{(r)}(x) - P_n^{(r)}(x)| \le C_2\big(n^{-1}\big(n^{-1} + \sqrt{1-x^2}\big)^{1-\lambda}\big)^{\alpha-r}. \tag{2.7.3}$$

Corollary 2.7.4. *For* $f \in C[-1,1]$ *and* $\alpha < r$ *the condition (2.7.1) or*

$$\delta_n(x)^{(\lambda-1)\alpha}|f(x) - P_n(x)| \le Cn^{-\alpha}$$

and

$$|(1-x^2)^{\lambda\alpha}\Delta_h^r f(x)| \le C_1 h^{\alpha} \tag{2.7.4}$$

are equivalent.

Corollary 2.7.4 is proved in a way similar to the treatment in [3].

Remark 2.7.5. *For* $\lambda = 0$, *Theorems 2.2.1 and 2.5.1 yield a somewhat different proof of the Timan, Dzjadyk, Freud and Brudnyi result. One should note, examining the proof for* $\lambda = 0$, *that we do not use theorems on* $\omega_\varphi^r(f,t)$. *We do, however, use the direction and methods of proofs developed in [4].*

References

[1] R. DeVore, $L_p[-1,1]$ *approximation by algebraic polynomials, linear spaces and approximation*, Eds. P. L. Butzer and B. Sz Nagy, Birkhauser Verlag, Basel, 1978, 397–406.

[2] Z. Ditzian, *On interpolation of* $L_p[a,b]$ *and weighted Sobolev spaces*, Pacific J. Math., **90** (1980), 307–323.

[3] Z. Ditzian, *Interpolation theorems on the rate of convergence of Bernstein polynomials*, Approximation Theory III, Ed. E. W. Cheney, Academic Press, 1980, 341–347.

[4] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, 1987.

[5] G. G. Lorentz, "Approximation of Functions," Holt, Rinehart and Winston, 1966. (Reprinted by Chelsea, 1988).

[6] V. P. Motornii, *Approximation of functions by algebraic polynomials in* L_p *metric*, Izv. Akad. Nauk. SSSR, **35** (1971), 874–899 (Russian).

[7] A. F. Timan, "Theory of Functions of Approximation of a Real Variable," English translation 1963, Pergamon Press, The MacMillan Co. (Russian original published in Moscow by Fizmatgiz in 1960).

The results of this chapter are joint work with Z. Ditzian and appeared in the Canadian Journal of Mathematics, **44**, 1992, pages 924-940.

CHAPTER III.

SHAPE PRESERVING POLYNOMIAL APPROXIMATION IN $C[-1,1]$

3.1. Introduction and the Main Result

When approximating a function, it is often important to preserve some of its features. For instance, it does not make sense to approximate a density function by elements that may take negative values. Here, we will approximate monotone and convex functions on $[-1,1]$ by polynomials and so, naturally, we prefer that the approximating polynomials will be monotone and convex respectively. Moreover, we will obtain an approximating polynomial that will coincide with f at ± 1.

We denote the class of polynomials of degree n by Π_n. The well-known Timan-type estimate for pointwise approximation of $f \in C[-1,1]$ by a polynomial P_n, $P_n \in \Pi_n$, is given by

$$|f(x) - P_n(x)| \leq C(r)\omega^r\big(f, n^{-1}(n^{-1} + \sqrt{1-x^2}\,)\big), \quad -1 \leq x \leq 1 \tag{3.1.1}$$

where

$$\omega^r(f,t) = \sup_{0 \leq h \leq t} \|\Delta_h^r f(x)\|_{C[-1,1]} \tag{3.1.2}$$

and

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^{r} \binom{r}{k} (-1)^k f(x + (\frac{r}{2} - k)h) & x \pm \frac{r}{2} h \in [-1,1] \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.3}$$

Restricted forms of this result appear in the standard texts on approximation by Lorentz [7, Chapter 5] and by Timan [8, Chapter 5]. The inequality (3.1.1) is Y. Brudnyi's extension of Timan's result. This, we trust will be the standard form of the Timan-type estimate (see [4, 7.1]) and will appear in that form in the much-awaited text on approximation by DeVore and Lorentz.

In 1985, it was shown by DeVore and Yu [2] that for a function $f \in C[-1,1]$ which is monotone, one can choose a monotone polynomial $P_n \in \Pi_n$, such that

$$|f(x) - P_n(x)| \leq C\omega^2\big(f, \frac{\sqrt{1-x^2}}{n}\big), \quad -1 \leq x \leq 1. \tag{3.1.4}$$

Later, in 1986, Leviatan [5] showed that when f is convex, one can choose a convex polynomial $P_n \in \Pi_n$ for which (3.1.4) is valid.

The norm estimate for the approximation of a function by polynomials was obtained by Ditzian and Totik [4, Chapter 7]. It was shown for $f \in L_p[-1,1]$, $1 \leq p \leq \infty$, that there exists $P_n \in \Pi_n$ such that

$$\|f - P_n\|_{L_p[-1,1]} \leq C(p,r)\omega_\varphi^r\big(f, n^{-1}\big)_p, \quad \varphi(x) = \sqrt{1-x^2} \tag{3.1.5}$$

where the new measure of smoothness $\omega_\psi^r(f,t)_p$ (see [4]) is given by

$$\omega_\psi^r(f,t)_p = \sup_{0<h\le t} \|\Delta_{h\psi}^r f\|_{L_p[-1,1]} \tag{3.1.6}$$

where ψ is a function of x and

$$\Delta_{h\psi}^r f(x) = \begin{cases} \sum_{k=0}^{r} \binom{r}{k} (-1)^k f\big(x + (\frac{r}{2} - k)h\psi(x)\big) & x \pm \frac{r}{2} h\psi(x) \in [-1,1] \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.7}$$

It is worth noting that for $0 < p < 1$, (3.1.5) was recently established by DeVore, Leviatan and Yu [1]. However, here we restrict our attention to $p = \infty$ and refer to (3.1.5) only for that case. We will write $\omega_\psi^r(f,t)$ for $\omega_\psi^r(f,t)_\infty$.

For $r = 2$, and $f \in C[-1,1]$ Leviatan [5, 6] proved that (3.1.5) (for $p = \infty$) is valid with $P_n \in \Pi_n$ preserving monotonicity or convexity.

For $f \in C[-1,1]$ and $0 \le \lambda \le 1$, it was recently shown by Ditzian and the author (see (2.1.6) in Chapter 2 or see [3]) that there exists $P_n \in \Pi_n$ such that

$$|f(x) - P_n(x)| \le C(r,\lambda)\omega_{\varphi^\lambda}^r\big(f, n^{-1}(n^{-1} + \sqrt{1-x^2})^{1-\lambda}\big), \quad -1 \le x \le 1 \tag{3.1.8}$$

and that result bridged the gap between (3.1.1) ($\lambda = 0$) and (3.1.5) ($\lambda = 1$).

We strengthen the above result for $r = 2$ in the following theorem which is the main result of this paper.

Theorem 3.1.1. *For $f(x) \in C[-1,1]$, $0 \le \lambda \le 1$ and $\varphi(x) = \sqrt{1-x^2}$, there*

exists $P_n \in \Pi_n$ *with* $P_n(\pm 1) = f(\pm 1)$ *such that*

$$|f(x) - P_n(x)| \le C(\lambda)\omega^2_{\varphi^\lambda}\big(f, n^{-1}\varphi(x)^{1-\lambda}\big), \quad -1 \le x \le 1 \tag{3.1.9}$$

with $C(\lambda)$ *independent of* n *and* f. *Moreover, if* f *is monotone or convex, or both,* P_n *can be chosen to be monotone or convex or both, respectively. The choice of* P_n *may be made independent of* λ.

3.2. Proof of the Main Result

In this section, we will prove the main result of the paper pending the proof of a Jackson-type inequality, which will be proved in stages in the next two sections. The proof here is constructive, that is, for $f \in C[-1,1]$, we will construct a sequence of polynomials $L_n(f,x)$ where L_n are linear operators on f such that

(3.2.1)

A. $L_n(f,x) \in \Pi_{4n}$,

B. $L_n(f,x)$ is monotone whenever $f(x)$ is montone,

C. $L_n(f,x)$ is convex whenever $f(x)$ is convex,

D. $L_n(f,\pm 1) = f(\pm 1)$,

E. $\|L_n(f,\cdot)\|_{C[-1,1]} \le C\|f\|_{C[-1,1]}$,

and

F. $|f(x) - L_n(f,x)| \le C(\lambda)\left(\frac{\varphi(x)^{1-\lambda}}{n}\right)^2 \|\varphi^{2\lambda} f''\|_\infty$ for $f \in C^2[-1,1]$.

It should be emphasized that we may use $L_n(f,x) \in \Pi_{4n}$ instead of $L_n(f,x) \in \Pi_n$ as P_n may be chosen to be $L_{[n/4]}(f,x)$ and

$$\omega^2_{\varphi^\lambda}(f,4t) \le C\omega^2_{\varphi^\lambda}(f,t).$$

The properties B, C, and D are the shape preserving properties.

We now use the equivalence (see [4, Chapter 2])

$$\omega^2_{\varphi^\lambda}(f,t) \sim K_{\varphi^\lambda,2}(f,t^2) \equiv \inf_{g \in C^2[-1,1]} (\|f-g\| + t^2\|\varphi^{2\lambda} g''\|). \tag{3.2.2}$$

Hence, there exists a function g_t satisfying

$$t^2\|\varphi^{2\lambda} g_t''\| \le C\omega^2_{\varphi^\lambda}(f,t) \tag{3.2.3}$$

and

$$\|f - g_t\| \le C\omega^2_{\varphi^\lambda}(f,t). \tag{3.2.4}$$

We choose $t = \varphi(x)^{1-\lambda}/n$ and apply E, F, and the linearity of $L_n(f,x)$ to

prove (3.1.9). Specifically,

$$|f(x) - L_n(f,x)| \le |f(x) - g_t(x)| + |g_t(x) - L_n(g_t,x)| + |L_n(f - g_t,x)|$$

$$\le (1 + \|L_n\|)\|f - g_t\| + C(\lambda)\big(\tfrac{\varphi(x)^{1-\lambda}}{n}\big)^2 \|\varphi^{2\lambda} g_t''\|$$

$$\le C\omega^2_{\varphi^\lambda}(f, \varphi(x)^{1-\lambda}/n).$$

Hence, the construction L_n satisfying A-F is sufficient to complete the proof.

Following [2] we define

$$L_n(f,x) = f(-1) + \sum_{j=0}^{n-1} s_j\big(R_j(x) - R_{j+1}(x)\big) \tag{3.2.5}$$

where

$$s_j = \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j}\,, \quad \xi_j = 1 - R_j(1) \tag{3.2.6}$$

and the polynomial $R_j(x)$ (depending of n) is defined by

$$R_j(x) = \int_{-1}^{x} r_j(u)\,du, \quad r_j(\cos t) = \int_{-\frac{n-j}{n}\pi}^{\frac{n-j}{n}\pi} K_n(t+u)\,du \tag{3.2.7}$$

where $K_n(t)$ is given by

$$K_n(t) = \lambda_n\Big(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\Big)^8, \quad \int_{-\pi}^{\pi} K_n(t)\,dt = 1. \tag{3.2.8}$$

The operator $L_n(f,x)$ is obviously linear. It is clear from (3.2.5–8) that $L_n(f,x)$ maps $C[-1,1]$ into Π_{4n}. As $R_0(x)=1+x$, $R_n(x)=0$ and $R_j(x)$ are strictly decreasing in j for $x>-1$, we have $-1=\xi_0<\cdots<\xi_n=1$. The identity (3.2.5) implies $L_n(f,\pm1)=f(\pm1)$. Also (3.2.5) can be rewritten as

$$L_n(f,x)=f(-1)+s_0R_0(x)+\sum_{j=1}^{n-1}(s_j-s_{j-1})R_j(x). \tag{3.2.9}$$

For a monotone $f(x)$, s_j is of fixed sign and since (3.2.7) implies that $R_j(x)-R_{j+1}(x)$ is increasing in x, (3.2.5) implies that $L_n(f,x)$ is monotone too. For a convex function f, s_j-s_{j-1} is of fixed sign and by (3.2.7) for $0<t<\pi$, we have (see [5, p.474])

$$\begin{aligned}
R_j''(x)=&r_j'(x)=\frac{-1}{\sin t}\frac{d}{dt}\int_{-\frac{n-j}{n}\pi}^{\frac{n-j}{n}\pi}K_n(t+u)\,du\\
=&\frac{-\lambda_n}{\sin t}\Big(\sin\frac{nt+(n-j)\pi}{2}\Big)^8\\
&\times\Big(\Big(\sin\big(\frac{t}{2}+\frac{n-j}{2n}\pi\big)\Big)^{-8}-\Big(\sin\big(\frac{t}{2}-\frac{n-j}{2n}\pi\big)\Big)^{-8}\Big)\\
\geq&0.
\end{aligned}$$

Hence, $L_n(f,x)$ is convex. In the last several lines, we summarized some of the known properties of $L_n(f,x)$ investigated by DeVore and Yu [2] and Leviatan [5]. DeVore and Yu [2, p.330] also proved the boundedness of $L_n(f,x)$ i.e. property E. Therefore, as we know properties A-E of $L_n(f,x)$, we will complete the proof of Theorem 3.1.1 when we prove the Jackson-type estimate, namely, property F of $L_n(f,x)$. This will be done in the next two section.

Remark 3.2.1. The polynomial $L_n(f,x)$ was constructed in a way that did not depend on λ, the monotonicity of f or the convexity of f. The construction here is independent of the construction in Chapter 2 or [3].

3.3. A Jackson-type Inequality for an Intermediate Operator

The goal of this section is to prove the Jackson-type estimate for an intermediary operator $S_n(f,x)$ defined below in (3.3.4).

We first recall some properties of the ξ_j's defined by (3.2.6) as shown by DeVore and Yu [2] (with slightly different ξ_j) (see also [5, *p.*473], Lemma A).

Lemma 3.3.1. *Let* $\delta_j = n^{-1}\sin\left(\frac{j}{n}\pi\right) + n^{-2}$, $j = 0,\dots,n$. *Then*

$$C^{-1}\delta_j \le \xi_{j+1} - \xi_j \le C\delta_j \quad \text{for} \quad j = 0,\dots,n-1 \tag{3.3.1}$$

$$C^{-1}\delta_j \le \delta_{j+1} \le C\delta_j \quad \text{for} \quad j = 0,\dots,n-1 \tag{3.3.2}$$

and

$$\xi_{j+1} - \xi_j \le C\sqrt{1-u^2}/n \quad \text{for} \quad \xi_j \le u \le \xi_{j+1} \quad \text{and} \quad j = 1,\dots,n-2. \tag{3.3.3}$$

We define the intermediary operator $S_n(f,x)$ by

$$S_n(f,x) = f(-1) + s_0(1+x) + \sum_{j=1}^{n-1}(s_j - s_{j-1})\varphi_j(x) \tag{3.3.4}$$

where $\varphi_j(x) = (x - \xi_j)_+$ and both ξ_i and s_j are given by (3.2.6).

Theorem 3.3.2. *For* $f \in C^2[-1,1]$, $0 \le \lambda \le 1$, $\varphi(x) = \sqrt{1-x^2}$ *and* $S_n(f,x)$ *defined by (3.3.4), we have*

$$|f(x) - S_n(f,x)| \le C\,\frac{\varphi(x)^{2-2\lambda}}{n^2}\,\|\varphi^{2\lambda} f''\|_{C[-1,1]}. \tag{3.3.5}$$

Proof: Following [6], we have for $\xi_j \le x \le \xi_{j+1}$, $0 \le j \le n-1$,

$$\begin{aligned} |f(x) - S_n(f,x)| &= \frac{(x-\xi_j)(\xi_{j+1}-x)}{\xi_{j+1}-\xi_j}\left|\frac{1}{\xi_{j+1}-x}\,(f(\xi_{j+1}) - f(x))\right. \\ &\qquad \left. - \frac{1}{x-\xi_j}\,(f(x) - f(\xi_j))\right| \\ &\le \frac{1}{\xi_{j+1}-\xi_j}\left[\int_{\xi_j}^{x} (u-\xi_j)(\xi_{j+1}-x)|f''(u)|\,du\right. \\ &\qquad \left. + \int_{x}^{\xi_{j+1}} (\xi_{j+1}-u)(x-\xi_i)|f''(u)|\,du\right]. \end{aligned} \tag{3.3.6}$$

The above implies

$$|f(x) - S_n(f,x)| \le (\xi_{j+1}-\xi_j)^2 \max_{\xi_j \le u \le \xi_{j+1}} \varphi(u)^{-2\lambda}\|\varphi^{2\lambda} f''\|_\infty \tag{3.3.7}$$

which, combined with (3.3.3) for $\xi_j \le u$, $x \le \xi_{j+1}$, yields (3.3.5) for $\xi_j \le x \le \xi_{j+1}$ and $1 \le j \le n-2$.

For $-1 \le x < \xi_1$, we use (3.3.6) to write

$$
\begin{aligned}
|f(x) - S_n(f,x)| &\le \frac{1}{1+\xi_1}\left[(\xi_1 - x)\int_{-1}^{x}(1+u)^{1-\lambda}\,du\right. \\
&\qquad \left. + (1+x)\int_{x}^{\xi_1}\frac{\xi_1 - u}{(1+u)^{\lambda}}\,du\right]\|\varphi^{2\lambda}f''\| \\
&\le \frac{1}{1+\xi_1}\left[(1+\xi_1)(1+x)^{2-\lambda}\right. \\
&\qquad \left. + (1+x)^{1-\lambda}(\xi_1 - x)^2\right]\|\varphi^{2\lambda}f''\|_\infty \\
&\le 2(1+\xi_1)\varphi(x)^{2-2\lambda}\|\varphi^{2\lambda}f''\|_\infty .
\end{aligned} \tag{3.3.8}
$$

Now (3.3.1) for $j = 0$ implies $1 + \xi_1 \le C_1 n^{-2}$, which yields (3.3.5) for $-1 \le x \le \xi_1$. The estimate (3.3.5) for $\xi_{n-1} \le x \le 1$ can be treated similarly. □

3.4. The Jackson-type Inequality for $L_n(f,x)$

In this section, we prove the Jackson inequality for $L_n(f,x)$. That is, we show that property F in (3.2.1) of Section 3.2 is valid and hence, we complete the proof of Theorem 3.1.1.

Following earlier works, we define

$$
d_{j,n}(t) = \max\left(1, n\ \mathrm{dist}(t, \{-\frac{j\pi}{n}, \frac{j\pi}{n}\})\right). \tag{3.4.1}
$$

The following lemma is crucial for our proof.

Lemma 3.4.1. *For* $0 \le \lambda \le 1$, $R_j(x)$ *defined by (3.2.7),* $\varphi_j(x) = (x-\xi_j)_+$, $d_{j,n}(t)$ *given by (3.4.1) and* $x = \cos t$, $0 \le t \le \pi$, *we have*

$$|\varphi_j(x) - R_j(x)| \le Cn^{-1}\varphi(x)^{2(1-\lambda)}(\sin \tfrac{j}{n}\pi)^{2\lambda-1}d_{n-j,n}(t)^{-5}. \tag{3.4.2}$$

Proof: We first use the estimate

$$|\varphi_j(x) - R_j(x)| \le Cn^{-1}\min\left\{\sin t, \sin\tfrac{n-j}{n}\pi\right\}d_{n-j,n}(t)^{-5} \tag{3.4.3}$$

obtained by combining [2, Lemma 5] and [6, Lemma 6]. For $2\lambda - 1 = \alpha \ge 0$, (3.4.3) implies (3.4.2) since $\min\{a,b\} \le a^{1-\alpha}b^{\alpha}$ for $0 \le a, b$ and $0 \le \alpha \le 1$. For $\sin t \ge \sin\frac{j}{n}\pi$, (3.4.3) implies (3.4.2) even if $2\lambda - 1 < 0$. Therefore, to complete the proof of (3.4.2), we assume that $-1 \le 2\lambda - 1 < 0$ and $\sin t < \sin\frac{n-j}{n}\pi$ in which case $0 \le t \le \pi$ implies that either $t \in \left(0, \min(\frac{n-j}{n}\pi, \frac{j}{n}\pi)\right) \equiv I_j'$ or $t \in \left(\max(\frac{n-j}{n}\pi, \frac{j}{n}\pi), \pi\right) \equiv I_j''$. Since $\varphi_j(\pm 1) = R_j(\pm 1)$, we have

$$\varphi_j(x) - R_j(x) = \int_{-1}^{x}\left(\varphi_j'(u) - R_j'(u)\right)du = -\int_{x}^{1}\left(\varphi_j'(u) - R_j'(u)\right)du \tag{3.4.4}$$

and we use $\int_{-1}^{x}$ and $\int_{x}^{1}$ for the estimate on I_j'' and I_j', respectively. As the proofs for $t \in I_j'$ and $t \in I_j''$ are essentially the same, we will give the estimate only for $t \in I_j'$. An important tool in the proof is the estimate by DeVore and Yu [2, (2.12b)]

$$|\varphi_j'(x) - R_j'(x)| \le Cd_{n-j,n}(t)^{-7}. \tag{3.4.5}$$

We write for $x = \cos t,\ t \in I_j'$ and $-1 \le 2\lambda - 1 < 0$,

$$
\begin{aligned}
|\varphi_j(x) - R_j(x)| &= \Big| \int_x^1 (\varphi_j'(u) - R_j'(u))du \Big| \\
&\le C \int_0^t d_{n-j,n}(v)^{-7} \sin v dv \\
&\le C\big(\sin \tfrac{j}{n}\pi\big)^{2\lambda-1} \int_0^t \sin v \big(\sin \tfrac{j}{n}\pi\big)^{1-2\lambda} d_{n-j,n}(v)^{-7} dv \\
&\equiv J_j,
\end{aligned}
$$

$$
\begin{aligned}
\big(\sin \tfrac{j}{n}\pi\big)^{1-2\lambda} &= \big(\sin \tfrac{n-j}{n}\,\pi\big)^{1-2\lambda} \\
&\le \big(\sin v + |v - \tfrac{n-j}{n}\,\pi|\big)^{1-2\lambda} \qquad (3.4.6) \\
&\le (\sin v)^{1-2\lambda} + |v - \tfrac{n-j}{n}\pi|^{1-2\lambda}.
\end{aligned}
$$

Hence, it is sufficient to estimate J_j' and J_j'' formed from J_j above by replacing $\big(\sin \frac{j}{n}\,\pi\big)^{1-2\lambda}$ by $(\sin v)^{1-2\lambda}$ and $|v - \frac{n-j}{n}\,\pi|^{1-2\lambda}$, respectively.

We observe that for $\alpha \ge \beta > 1$ and $t < \frac{n-j}{n}\,\pi$,

$$
\begin{aligned}
\int_0^t d_{n-j,n}(v)^{-\alpha} dv &\le C(\alpha) n^{-1} d_{n-j,n}(t)^{-\alpha+1} \qquad (3.4.7) \\
&\le C(\alpha) n^{-1} d_{n-j,n}(t)^{-\beta+1}.
\end{aligned}
$$

We now write, using (3.4.7) with $\alpha = 7$ and $\beta = 6$,

$$J_j' \le C\big(\sin \tfrac{j}{n}\pi\big)^{2\lambda-1} \int_0^t (\sin v)^{2-2\lambda} d_{n-j,n}(v)^{-7}\, dv$$

$$\le C_1\big(\sin \tfrac{n-j}{n}\pi\big)^{2\lambda-1} (\sin t)^{2-2\lambda} \int_0^t d_{n-j,n}(v)^{-7}\, dv$$

$$\le C_1 n^{-1}\big(\sin \tfrac{n-j}{n}\pi\big)^{2\lambda-1} (\sin t)^{2-2\lambda} d_{n-j,n}(t)^{-5}.$$

We now use (3.4.7) with $\alpha = 7 + 2\lambda - 1$ and $\beta = 6$ and the inequality

$$|v - \tfrac{n-j}{n}\pi| \le n^{-1} d_{n-j,n}(v)$$

to estimate J_j'' by

$$J_j'' = C\big(\sin \tfrac{j}{n}\pi\big)^{2\lambda-1} \int_0^t |v - \tfrac{n-j}{n}\pi|^{1-2\lambda} d_{n-j,n}(v)^{-7} \sin v\, dv$$

$$= C\big(\sin \tfrac{j}{n}\pi\big)^{2\lambda-1} n^{2\lambda-2} d_{n-j,n}(t)^{-5} \sin t.$$

On the other hand an elementary estimate of J_j'' yields

$$J_j'' \le C\big(\sin \tfrac{j}{n}\pi\big)^{2\lambda-1} t n^{2\lambda-1} d_{n-j,n}(t)^{-5} \sin t.$$

Combining the above, we have

$$J_j'' \le C\big(\sin \tfrac{j}{n}\,\pi\big)^{2\lambda-1} (\sin t) n^{2\lambda-1} d_{n-j,n}(t)^{-5} \min\Big(t, \frac{1}{n}\Big)$$

and since $\min(t, n^{-1}) \le t^{1-2\lambda} n^{-2\lambda}$ (for $0 \le 1 - 2\lambda \le 1$) and $t \le \frac{\pi}{2} \sin t$ (for $t \in [0, \frac{\pi}{2}]$), the proof is complete. □

We now prove the main result of this section.

Theorem 3.4.2. *For* $f \in C^2[-1,1]$, $0 \le \lambda \le 1$, $\varphi(x) = \sqrt{1-x^2}$ *and* $L_n(f,x)$ *given by (3.2.5), we have*

$$|f(x) - L_n(f,x)| \le C\,\frac{\varphi(x)^{2-2\lambda}}{n^2}\,\|\varphi^{2\lambda} f''\|_{C[-1,1]}. \tag{3.4.8}$$

Proof: Applying Theorem 3.3.2, it is sufficient to show that $|L_n(f,x) - S_n(f,x)|$ is bounded by the expression on the right hand side of (3.4.8). By virtue of (3.2.9), (3.3.4) and Lemma 3.4.1, we write

$$|L_n(f,x) - S_n(f,x)| \le \sum_{j=1}^{n-1} |s_j - s_{j-1}|\,|R_j(x) - \varphi_j(x)|$$

$$\le Cn^{-1}\varphi(x)^{2(1-\lambda)} \sum_{j=1}^{n-1} |s_j - s_{j-1}| (\sin \tfrac{j}{n}\,\pi)^{2\lambda-1} d_{n-j,n}(t)^{-5} \tag{3.4.9}$$

where $x = \cos\, t$.

We now use the Taylor formula to estimate $|s_j - s_{j-1}|$ by

$$|s_j - s_{j-1}| = \left| \frac{f(\xi_{j+1}) - f(\xi_j)}{\xi_{j+1} - \xi_j} - \frac{f(\xi_j) - f(\xi_{j-1})}{\xi_j - \xi_{j-1}} \right|$$

$$\le \int_{\xi_{j-1}}^{\xi_j} \frac{u - \xi_{j-1}}{\xi_j - \xi_{j-1}}\,|f''(u)|\,du + \int_{\xi_j}^{\xi_{j+1}} \frac{\xi_{j+1} - u}{\xi_{j+1} - \xi_j} |f''(u)|\,du.$$

For $j = 2, \ldots, n-2$, we use Lemma 3.3.1 (first (3.3.3) then (3.3.1) and

(3.3.2)) to obtain

$$|s_j - s_{j-1}| \le \Big[|\xi_j - \xi_{j-1}| \max_{\xi_{j-1} \le u \le \xi_j} \varphi(u)^{-2\lambda} + |\xi_{j+1} - \xi_j| \max_{\xi_j \le u \le \xi_{j+1}} \varphi(u)^{-2\lambda}\Big] \|\varphi^{2\lambda} f''\|$$

$$\le Cn^{-2\lambda}\big[|\xi_j - \xi_{j-1}|^{1-2\lambda} + |\xi_{j+1} - \xi_j|^{1-2\lambda}\big]\|\varphi^{2\lambda} f''\|$$

$$\le C_1 n^{-2\lambda}\delta_j^{1-2\lambda} \le C_2 n^{-1}\big(\sin \tfrac{j}{n}\pi\big)^{1-2\lambda}.$$

For the last inequality we used the estimate $n^{-1}\sin\frac{j}{n}\pi \le \delta_j \le Cn^{-1}\sin\frac{j}{n}\pi$ for $1 \le j \le n-1$.

For $j = 1$ (and similarly for $j = n-1$), we apply Lemma 3.3.1 to write

$$|s_1 - s_0| \le \left[\frac{1}{1+\xi_1}\int_{-1}^{\xi_1}(u+1)^{1-\lambda}du + \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2}(\xi_2 - u)\varphi(u)^{-2\lambda}du\right]\|\varphi^{2\lambda} f''\|$$

$$\le C_3\left[(\xi_1+1)^{1-\lambda} + n^{-1}\Big(\sin\tfrac{1}{n}\pi\Big)^{1-2\lambda}\right]\|\varphi^{2\lambda} f''\|.$$

Using Lemma 3.3.1 once more, we obtain

$$|s_1 - s_0| \le C_4 n^{-1}\big(\sin\tfrac{1}{n}\pi\big)^{1-2\lambda}\|\varphi^{2\lambda} f''\|.$$

Combining the estimate which we achieved, i.e.,

(3.4.10) $$|s_j - s_{j-1}| \le Cn^{-1}\big(\sin\tfrac{j}{n}\pi\big)^{1-2\lambda}\|\varphi^{2\lambda} f''\|$$

with (3.4.9), we complete the proof recalling (2.24) of [2], i.e.,

$$\sum_{j=1}^{n-1} d_{n-j,n}(t)^{-5} \leq A \tag{3.4.11}$$

with A independent of t and n. □

References

[1] R. A. DeVore, D. Leviatan and X.M. Yu, *Polynomial approximation in L_p $(0 < p < 1)$*, Constr. Approx. **8** (1992), 187-201.

[2] R. A. DeVore and X.M. Yu, *Pointwise estimates for monotone polynomial approximation,* Constr. Approx. **1** (1985), 323-331.

[3] Z. Ditzian and D. Jiang, *Approximation of functions by polynomials in $C[-1,1]$*, Canad. J. Math. **44** 5, (1992), 924-940.

[4] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, 1987.

[5] D. Leviatan, *Pointwise estimates for convex polynomial approximation,* Proc. Amer. Math. Soc. **98** (1986), 471-474.

[6] D. Leviatan, *Monotone and comonotone polynomial approximation revisited,* J. Approx. Theory **53** (1988), 1-16.

[7] G. G. Lorentz, "Approximation of Functions," Holt, Rinehart and Winston, 1966 (Reprinted by Chelsea, 1988).

[8] A. F. Timan, "Theory of Approximation of Functions of a Real Variable," English translation 1963 Pergamon Press. (Russian original published by Fizmatgiz, 1960.)

The results of this chapter are joint work with Z. Ditzian and D. Leviatan and appeared in the Mathematical Proceedings of the Cambridge Philosophical Society, **112**, 1992, pages 309-316.

CHAPTER IV.

SIMULTANEOUS POLYNOMIAL APPROXIMATION

4.1. Introduction

It is known (see (2.1.6) in Chapter 2 or see [5]) that for $f \in C[-1,1]$, $0 \leq \lambda \leq 1$ and an integer r there exists a polynomial $P_n \in \Pi_n$, where Π_n is the class of polynomials of degree n, such that

$$|f(x) - P_n(x)| \leq C\omega^r_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}) \tag{4.1.1}$$

where $\varphi(x) = \sqrt{1-x^2}$, $\delta_n(x) = n^{-1} + \varphi(x)$ and $\omega^r_\psi(f,t)$ is given by

$$\omega^r_\psi(f,t) = \sup_{0<h<t} \sup_{x \in I} |\Delta^r_{h\psi(x)} f(x)|, \quad I = [-1,1]. \tag{4.1.2}$$

This measure of smoothness related to the step-weight function ψ was introduced and investigated in [7]. (For the case treated here $\omega^r_\psi(f,t)$ was already introduced in [4, §3].) We note that when $\lambda = 0$, (4.1.1) yields the Timan-type estimate for polynomial approximation and when $\lambda = 1$, (4.1.1) yields the norm estimate (only for $L_\infty[-1,1]$) for best polynomial approximation (see [7, Chapter 7] for the result in $L_p[-1,1]$, $1 \leq p \leq \infty$).

For $r = 2$ it was shown (see Theorem 3.1.1 in Chapter 3 or see [6]) that $P_n \in \Pi_n$ can be found such that

$$|f(x) - P_n(x)| \le C\omega^2_{\varphi^\lambda}(f, n^{-1}\varphi(x)^{1-\lambda}). \tag{4.1.3}$$

In this paper we investigate the situation in which $f^{(s)} \in C[-1,1]$. We will show:

Theorem 4.1.1. *Suppose s, r and k are integers, $s \ge 0$, $r \ge 1$, $0 \le \lambda \le 1$, $f^{(s)}(x) \in C[-1,1]$, and $\varphi(x) = \sqrt{1-x^2}$. Then, for $n \ge s+1$ there exists $P_n \in \Pi_n$ for which*

$$|f^{(j)}(x) - P_n^{(j)}(x)| \le C\big(n^{-1}\varphi(x)\big)^{s-j}\omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big), \quad 0 \le j \le s, \tag{4.1.4}$$

and

$$|P_n^{(s+k)}(x)| \le Cn^k\delta_n(x)^{-k}\omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big), \quad k \ge r. \tag{4.1.5}$$

The construction used for Theorem 4.1.1 can be refined to prove the following sharper result for $r = 1, 2$.

Theorem 4.1.2. *Under the assumptions of Theorem 4.1.1, for $n \ge s+1$ there exists a polynomial $P_n \in \Pi_n$ such that*

$$|f^{(j)}(x) - P_n^{(j)}(x)| \le C\big(n^{-1}\varphi(x)\big)^{s-j}\omega^\ell_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\big) \tag{4.1.6}$$

for $\ell = 1, 2$ *and* $0 \le j \le s$, *and*

$$|P_n^{(s+k)}(x)| \le Cn^k \delta_n(x)^{-k} \omega_{\varphi^\lambda}^\ell \left(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\right), \tag{4.1.7}$$

for $k \ge \ell$.

Remark 4.1.8. In the preceding theorem it is sufficient to prove the result for $\ell = 2$ in (4.1.6) and (4.1.7) since

$$\omega_{\varphi^\lambda}^2(g,t) \le M\omega_{\varphi^\lambda}(g,t) \quad \text{for} \quad 0 \le t \le t_0 \tag{4.1.8}$$

(see [7, Theorem 4.1.3]). The remaining case, i.e., $k = \ell = 1$ can be treated similarly (see also the remark after (4.3.13)).

For the special case $\lambda = 0$ a great part (but not all) of the above theorems were proved in a series of articles spread out from the early fifties to the last decade. For $\lambda > 0$ the results are new. The earlier investigation of this type of estimate seems to go back to Timan [14] in 1951 and his work was extended in the sixties by Trigub [15] and Brudnyĭ [1] who achieved a special case of (4.1.4) for $\lambda = 0$. Improving the estimates by replacing $\delta_n(x)$ by $\varphi(x)$ is due to Teljakovskiĭ [13] and Gopengauz [10] who got (4.1.6) with $\ell = 1$ and $\lambda = 0$. Gopengauz [11] made an attempt at (4.1.6) for $\lambda = 0$ and other ℓ but he claimed that the result is valid for all ℓ which is known to be incorrect for $\ell \ge 3$. It seems that the first proof of (4.1.6) for $\ell = 2$ (under the conditions $\lambda = 0$ and $s = 0$) is due to DeVore [2], [3]. This result was further extended by Gonska and Hinnemann [9] who proved (4.1.6) for $\lambda = 0$, $\ell = 2$ and $0 \le j \le s - 2$.

4.2. Some Preliminary Lemmas

We will prove here some lemmas needed for the proof of the main result.

Lemma 4.2.1. *Suppose* $P_n \in \Pi_n$, $m \geq 0$, $0 \leq \lambda \leq 1$ *and* $\delta_n(x) = \frac{1}{n} + \sqrt{1-x^2}$ *and that the increasing function* $\omega(t)$ *satisfies*

$$0 < \omega(\mu t) \leq M_r(\mu^r + 1)\omega(t) \tag{4.2.1}$$

for some positive integer r. *Then*

$$|P_n(x)| \leq A\big(n^{-1}\delta_n(x)\big)^m \omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \quad -1 \leq x \leq 1, \tag{4.2.2}$$

implies for $k \geq 0$

$$|P_n^{(k)}(x)| \leq C\big(n^{-1}\delta_n(x)\big)^{m-k} \omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \quad -1 \leq x \leq 1, \tag{4.2.3}$$

where C *depends on* k, m, λ, r, A *and* M_r *but not on* x *or* n.

Proof: For $k \geq m + r(1-\lambda)$ (4.2.3) follows from Theorem 2.4.1 in Chapter 2 (or see [5]). We set $k_0 = [m + r(1-\lambda)] + 1$ and prove the result for $0 < k < k_0$ using the inequality

$$\|g^{(k)}\|_{L_\infty[c,d]} \leq Mh^{-k}\|g\|_{L_\infty[c,d]} + h^{k_0-k}\|g^{(k_0)}\|_{L_\infty[c,d]}, \quad 0 < h \leq d-c, \tag{4.2.4}$$

with $M = M(k_0)$ independent of g, $[c,d]$ and $0 < h \leq d-c$. The inequality is well-known without the independence of M and was proved for $g \in L_p[c,d]$ and

$0 < h \le d-c$ in [4, Lemma 2.1]. Although in the statement of [4, Lemma 2.1] $d-c$ appears rather than h, the proof applies to any h satisfying $0 < h \le d-c$. (We remark that Lemma 2.1 of [4] has been used extensively and always with $h = d-c$ but that in fact the stronger version, (4.2.4) was proved there and is needed here.) We now prove (4.2.3) in a collection of intervals, the union of which covers $[-1,1]$. As the result is symmetric, we will show it only for $[-1,0]$. We first choose $x \in [c,d] = [-1, -1+n^{-2}]$ and $h = n^{-2}$. For such an x,

$$\delta_n(x) \sim n^{-1} \quad \text{and} \quad \omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \sim \omega(n^{-2+\lambda})$$

and hence, (4.2.3) follows from (4.2.4) with $g = P_n$ in that interval. We now choose $x \in [c,d] = [-1+n^{-2}2^{\ell},\ -1+n^{-2}2^{\ell+1}]$, $0 \le \ell$, $2^{\ell+1} \le n^2$ and $h = n^{-2}2^{\ell/2}$. For x in such an interval

$$\delta_n(c) \le \delta_n(x) \le \delta_n(d) \le \sqrt{2}\,\delta_n(c), \quad h \le n^{-1}\delta_n(c) \le 2h$$

and

$$\begin{aligned}\omega\big(n^{-1}\delta_n(c)^{1-\lambda}\big) &\le \omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \le \omega\big(n^{-1}\delta_n(d)^{1-\lambda}\big) \\ &\le M_r(2^{r(1-\lambda)/2}+1)\omega\big(n^{-1}\delta_n(c)^{1-\lambda}\big).\end{aligned}$$

Therefore, (4.2.3) follows from (4.2.4) with $g = P_n$ for $x \in [-1+n^{-2}2^{\ell},\ -1+n^{-2}2^{\ell+1}]$ and those ℓ. It is sufficient now to estimate (4.2.3) for $x \in [c,d] = [-1+n^{-2}2^{\ell_0}, 0]$ $(2^{\ell_0+1} \le n^2 < 2^{\ell_0+2})$. We choose $h = n^{-1}$ and recall that for

$x \in [c,d]$

$$\delta_n(x) \sim \delta_n(n^{-1}) \sim 1 \quad \text{and} \quad \omega\big(n^{-1}\delta(x)^{1-\lambda}\big) \sim \omega(n^{-1})$$

from which (4.2.3) follows for $[c,d] = [-1+n^{-2}2^{\ell_0}, 0]$. □

We now prove what will constitute the crucial step in the inductive proof of our main result.

Lemma 4.2.2. *Suppose* ℓ *is an integer,* $\ell \geq -1$, $0 \leq \lambda \leq 1$, $\omega(t)$ *is an increasing function satisfying (4.2.1) for some integer* r, $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$ *and* g_n *satisfies*

$$|g_n'(x)| \leq C\big(n^{-1}\delta_n(x)\big)^{\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big). \tag{4.2.5}$$

Then there exists a polynomial $Q_n \in \Pi_n$ *satisfying*

$$|g_n(x) - Q_n(x)| \leq M\big(n^{-1}\delta_n(x)\big)^{\ell+1}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \tag{4.2.6}$$

and

$$|Q_n'(x)| \leq M\big(n^{-1}\delta_n(x)\big)^{\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \tag{4.2.7}$$

with $M \equiv M(C,\ell,r,\lambda)$ *independent of* n *and* g_n.

Proof: We construct the algebraic polynomial $Q_n \in \Pi_n$ by

$$Q_n(x) \equiv \int_{-\pi}^{\pi} g_n\big(\cos((\arccos\, x) - t)\big)T_n(t)dt \tag{4.2.8}$$

where $T_n(t)$ is the trigonometric polynomial of order n given by

$$T_n(t) \equiv L_{[n/m],m}(t), \quad L_{k,m}(t) = \lambda_k \Big(\frac{\sin\ kt/2}{\sin\ t/2}\Big)^{2m}$$

(4.2.9)

$$\int_{-\pi}^{\pi} L_{k,m}(t)dt = 1 \quad \text{and} \quad m = \ell + r + 2.$$

To prove (4.2.6) we write

$$I_n(x) = |g_n(x) - Q_n(x)| = |\int_{-\pi}^{\pi} [\int_{\cos\big((\arccos\ x)-t\big)}^{x} g_n'(u)du]T_n(t)dt|.$$

We denote the interval between $\cos\big((\arccos\ x) - t\big)$ and x by $J(x,t)$ and its length by $m\big(J(x,t)\big)$. Using straightforward computation, we have (see [7, p.80])

$$m\big(J(x,\pm t)\big) = |x - \cos\big((\arccos\ x) \pm t\big)|$$

(4.2.10)

$$\leq \Delta_{1/t}(x) \equiv |t|\ (|t| + \sqrt{1-x^2}\ ).$$

Hence,

$$I_n(x) \leq \int_{-\pi}^{\pi} \Big\{\frac{1}{m\big(J(x,t)\big)} \int_{J(x,t)} \Delta_{1/t}(x)|g_n'(u)|du\Big\}T_n(t)dt$$

$$\leq Cn^{-\ell} \int_{-\pi}^{\pi} \Big\{\frac{1}{m\big(J(x,t)\big)} \int_{J(x,t)} \Delta_{1/t}(x)\delta_n(u)^{\ell}\omega\big(n^{-1}\delta_n(u)^{1-\lambda}\big)du\Big\}T_n(t)dt.$$

Obviously,

$$\Delta_{1/t}(x) \leq |t|(n|t| + 1)\delta_n(x).$$

Using (4.2.1), we have

$$\omega\big(f, n^{-1}\delta_n(u)^{1-\lambda}\big) \leq C_1\Big(\big(\frac{\delta_n(u)}{\delta_n(x)}\big)^{r(1-\lambda)} + 1\Big)\omega\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

For $u \in J(x,t)$ we have $|x-u| \le \min\big(\Delta_{1/t}(x), \Delta_{1/t}(u)\big)$ and hence

$$\delta_n(u) \le 9\ \max(n^2t^2, 1)\delta_n(x) \quad \text{for} \quad u \in J(x,t).$$

(See [7, *pp.*80-81] where the results were proved for $|x-u| \le \min\big(\Delta_{1/t}(x), \Delta_{1/t}(u)\big)$ and hence the roles of u and x are symmetric.) Therefore, we have

$$I_n(x) \le C_2 n^{-\ell}\delta_n(x)^{\ell+1}\omega\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)$$
$$\times \int_{-\pi}^{\pi} |t|(n|t|+1)(n^2t^2+1)^{\ell}\big((n^2t^2+1)^{r(1-\lambda)}+1\big)T_n(t)dt.$$

We now recall

$$\int_{-\pi}^{\pi} |t|^{\gamma} L_{n,m}(t)dt \le Ln^{-\gamma} \quad \text{for} \quad \gamma < 2m-1$$

and hence, for $m \ge r+\ell+2$

$$I_n(x) \le C_3\big(n^{-1}\delta_n(x)\big)^{\ell+1}\omega^r\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

We now follow [8, *pp.*127-128] to prove (4.2.7). Using (4.2.8), $T_n(u) = T_n(-u)$ and [8, *p.*128], we have

$$Q_n'(x) = -\int_0^{\pi} g_n'(\cos\xi)2\sin u\, T_n'(u)du,$$

with $\cos\xi \in J(x,u)$. Following [8, *p.*128], we have

$$|Q_n'(x)| \le C_4 \int_0^{\pi} |g_n'(\cos\xi)| L_{[n/m],m-1}(u)du. \tag{4.2.11}$$

Earlier considerations in this proof yield

$$|g_n'(\cos\xi)| \le C\big(n^{-1}\delta_n(\cos\xi)\big)^{\ell}\omega\big(n^{-1}\delta_n(\cos\xi)^{1-\lambda}\big)$$

$$\le C_5\big(n^{-1}\delta_n(x)\big)^{\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big)$$

$$\times\big((nu)^2+1\big)^{\ell}\big(((nu)^2+1)^{r(1-\lambda)}+1\big).$$

Since

$$\int_{-\pi}^{\pi}|t|^{\gamma}L_{[n/m],m-1}(t)dt \le L_1 n^{-\gamma} \quad \text{for} \quad \gamma < 2m-3,$$

the result is valid if $2\ell + 2r < 2m-3$ which is evident from (4.2.9). □

Remark 4.2.3. *Examining the proof of Lemma 4.2.2, we can replace the integer* ℓ *satisfying* $\ell \ge -1$ *by any real number satisfying* $\beta \ge -1$. *In this case the choice* $m = [\beta] + r + 3$ *in (4.2.9) will do.*

Lemma 4.2.4 (Trigub). *For any two integers* $\ell > 0$ *and* $m > 0$ *we have a polynomial* $P_n \in \Pi_n$ *such that*

$$\|(1-x^2)^{\ell} - (1-x^2)^{\ell+m}P_n(x)\|_{C[-1,1]} \le C(\ell,m)n^{-2\ell}. \tag{4.2.12}$$

This lemma is a corollary of Lemma $2''$ of [15]. The proof in [15] is computational and long (via Lemma 2 and Lemma $2'$ there). It is amazing that this best estimate (up to a constant) was achieved by Trigub in 1962 without many modern

tools such as ω_φ^r and others. We will give a different, more accessible and, we hope, more transparent, proof below.

Proof of Lemma 4.2.4: For $f(x) = \sqrt{1-x^2}$ we have $\omega_\varphi(f,t) \sim t$ (see [7, $p.109(1)$]). Hence, (see [7, Theorem 7.2.1]) there is $Q_n \in \Pi_n$ for which

$$\|\sqrt{1-x^2} - Q_n(x)\|_{C[-1,1]} \leq Cn^{-1}.$$

Writing

$$Q_{n,1}(x) = Q_n(x) - \frac{(1-x)}{2} Q_n(-1) - \frac{(1+x)}{2} Q_n(1),$$

we have $Q_{n,1}(\pm 1) = 0$, $Q_{n,1}(x) = (1-x^2)Q_{n,2}(x)$ and therefore,

$$\|\sqrt{1-x^2} - (1-x^2)Q_{n,2}(x)\|_{C[-1,1]} \leq 3Cn^{-1}. \tag{4.2.13}$$

For $-1+n^{-2} \leq x \leq 1-n^{-2}$

$$|(1-x^2)^{1/2}Q_{n,2}(x)| \leq 1 + 3Cn^{-1}(1-x^2)^{-1/2} \leq 1+3C.$$

Using [7, Theorem 8.4.8], we have

$$\|(1-x^2)^{1/2}Q_{n,2}(x)\|_{C[-1,1]} \leq A\|(1-x^2)^{1/2}Q_{n,2}(x)\|_{C[-1+\frac{1}{n^2},1-\frac{1}{n^2}]}$$

$$\leq A(1+3C).$$

We now use (4.2.13) $2m$ times to obtain

$$\begin{aligned}
&\|\sqrt{1-x^2}-\sqrt{1-x^2}\,(1-x^2)^m Q_{n,2}^{2m}(x)\| \\
&\qquad \le \|\sqrt{1-x^2}-(1-x^2)^m Q_{n,2}^{2m-1}(x)\| \\
&\qquad\qquad + \|(\sqrt{1-x^2}-(1-x^2)Q_{n,2}(x))(1-x^2)^{m-1/2}Q_{n,2}^{2m-1}(x)\| \\
&\qquad \le 3Cn^{-1}\big(1+\cdots+\quad (A(1+3C)^{2m-1})\big).
\end{aligned}$$

The desired estimate is that of $\left(\sqrt{1-x^2}-\sqrt{1-x^2}\,(1-x^2)^m Q_{n,2}^{2m}(x)\right)^{2\ell}$. The fact that the polynomial

$$P(x)=\sum_{j=1}^{2\ell}(-1)^{j-1}\binom{2\ell}{j}(1-x^2)^{m(j-1)}Q_{n,2}^{2mj}(x)$$

is not of degree n but rather of degree $4\ell m(n-1)-2m$ does not matter since ℓ and m are fixed. □

4.3. The Main Result

In this section we will prove Theorems 4.1.1 and 4.1.2 which, together, constitute our main result. It turns out that it is advantageous to prove both theorems simultaneously.

Proof of Theorems 4.1.1 and 4.1.2: We use Theorem 2.2.1 in Chapter 2 (or see Theorem 2.1 of [5]) applied to $f^{(s)}(x)$ to ascertain the existence of a polynomial

$q_n \in \Pi_n$ satisfying

$$|f^{(s)}(x) - q_n(x)| \le C\omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big). \tag{4.3.1}$$

We also use Theorem 3.1.1 in Chapter 3 (or see Theorem 1.1 of [6]) to obtain a polynomial $q_n(x) \in \Pi_n$ satisfying

$$|f^{(s)}(x) - q_n(x)| \le C\omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\big). \tag{4.3.2}$$

In the construction of P_n it will make no difference whether we choose $q_n(x)$ to satisfy (4.3.1) or (4.3.2) as

$$\omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\big) \le \omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

The difference will emerge in the proofs of (4.1.4), (4.1.5), (4.1.6) and (4.1.7) as a result of that construction. Our construction will yield a polynomial P_n such that $P_n \in \Pi_{Ln}$ with some fixed integer L. This will not change the result as, in the right hand side of (4.1.4), ..., (4.1.7), replacing n by Ln would just change the constant.

With no loss of generality we may assume that

$$f(x) = \int_0^x \int_0^{u_{s-1}} \cdots \int_0^{u_1} f^{(s)}(u)du$$

as $f(x)$ may differ from the above only by a polynomial of degree $s-1$ which can be absorbed by $P_n(x)$.

We recall from [7, Theorem 4.1.2] that $\omega^r_{\varphi^\lambda}(f,t)$ satisfies condition (4.2.1). This follows from the equivalence

$$\omega^r_{\varphi^\lambda}(f,t) \sim K_{\varphi^\lambda,r}(f,t^r)$$

where

$$K_{\varphi^\lambda,r}(f,t^r) = \inf_g \left(\|f-g\| + t^r \|\varphi^{\lambda r} g^{(r)}\| \right),$$

and the basic properties of any K-functional, i.e. $K(f,t)$ is increasing and $K(f,t^r) \le a^{-r} K(f,(at)^r)$ for $0 < a < 1$. Hence, using Lemma 4.2.2 with $\ell = 0$ and

$$g_n(x) \equiv \int_0^x (f^{(s)}(u) - q_n(u))du = f^{(s-1)}(x) - \int_0^x q_n(u)du,$$

we have a polynomial $q_{n,1}(u) \in \Pi_n$ such that

$$|f^{(s-1)}(x) - \int_0^x q_n(u)du - q_{n,1}(x)| \le C_1 \big(n^{-1}\delta_n(x)\big) \omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big)$$

and

$$|q'_{n,1}(x)| \le C_1 \omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta(x)^{1-\lambda}\big).$$

This implies

$$|f^{(s)}(x) - q_n(x) - q'_{n,1}(x)| \le (C + C_1)\omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

We now proceed (in case $s > 1$), using Lemma 4.2.2, with $\ell = 1$ and

$$g_n(x) = f^{(s-2)}(x) - \int_0^x \int_0^{u_1} q_n(u)du\, du_1 - \int_0^x q_{n,1}(u)du.$$

In general we assume by induction that for some $j < s$ we have already constructed $q_{n,0}, q_{n,1}(x), \dots, q_{n,j}(x) \in \Pi_n \quad (q_{n,0} \equiv q_n)$ such that for $0 \le i \le j$

$$
\begin{aligned}
&|f^{(s-i)}(x) - \int_0^x \int_0^{u_{i-1}} \cdots \int_0^{u_1} q_n(u)du\, du_1 \dots du_{i-1} - \dots \\
&\qquad - \int_0^x q_{n,i-1}(u)du - q_{n,i}(x) - \cdots - q_{n,j}^{(j-i)}(x)| \\
&\qquad \le C\big(n^{-1}\delta_n(x)\big)^i \omega_{\varphi^\lambda}^r\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big).
\end{aligned} \tag{4.3.3}
$$

We now use Lemma 4.2.2 with $\ell = j$ and with $g_n(x)$ given by

$$
g_n(x) = f^{(s-j-1)}(x) - \int_0^x \int_0^{u_j} \cdots \int_0^{u_1} q_n(u)du\, du_1 \dots du_j - \cdots - \int_0^x q_{n,j}(u)du
$$

to obtain $q_{n,j+1}$ that satisfies (4.3.3) for $f^{(s-j-1)}$ and $i = j+1$. Moreover, Lemma 4.2.2 implies

$$
|q'_{n,j+1}(x)| \le M\big(n^{-1}\delta_n(x)\big)^j \omega_{\varphi^\lambda}^r\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big).
$$

Using Lemma 4.2.1 with $m = j$ and $P_n(x) = q'_{n,j+1}(x)$, we have

$$
|q_{n,j+1}^{(k+1)}(x)| \le M_1\big(n^{-1}\delta_n(x)\big)^{j-k} \omega_{\varphi^\lambda}^r\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big). \tag{4.3.4}
$$

Hence, (4.3.3) is valid for $j+1$ and $0 \le i \le j+1$ and by induction, we construct $q_n, q_{n,1}, \dots, q_{n,s}$. We define

$$
Q_{n,0}(x) = \int_0^x \int_0^{u_{s-1}} \cdots \int_0^{u_1} q_n(x)du \tag{4.3.5}
$$

and

$$(4.3.6) \qquad Q_{n,1}(x) = \int_0^x \int_0^{u_{s-2}} \cdots \int_0^{u_1} q_{n,1}(x)du\, du_1 \dots du_{s-2} + \cdots + q_{n,s}(x).$$

The construction above yields for $0 \le i \le s$,

$$(4.3.7) \qquad \begin{aligned} &|f^{(i)}(x) - Q_{n,0}^{(i)}(x) - Q_{n,1}^{(i)}(x)| \\ &\qquad \le A\big(n^{-1}\delta_n(x)\big)^{s-i}\omega_{\varphi^\lambda}^r\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big), \end{aligned}$$

$Q_{n,0}^{(s)}(x) = q_n(x)$ for $q_n(x)$ given in (4.3.1) or (4.3.2) and

$$(4.3.8) \qquad |Q_{n,1}^{(s+k)}(x)| \le A(k)\big(n^{-1}\delta_n(x)\big)^{-k}\omega_{\varphi^\lambda}^r\big(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\big)$$

$$\text{for} \quad k = 0, 1, \dots .$$

We follow Gopengauz [10] (see also [12]) with some changes and construct a polynomial $Q_{n,2}(x)$,

$$(4.3.9) \qquad Q_{n,2}(x) = \sum_{i=0}^{s} (1-x^2)^i\big(\alpha_i(1-x) + \beta_i(1+x)\big),$$

such that

$$(4.3.10) \qquad Q_{n,2}^{(i)}(\pm 1) = f^{(i)}(\pm 1) - Q_{n,0}^{(i)}(\pm 1) - Q_{n,1}^{(i)}(\pm 1) \quad \text{for} \quad 0 \le i \le s.$$

The fact that n^{-2s+2i} is increasing in i,

$$|f^{(i)}(\pm 1) - Q_{n,0}^{(i)}(\pm 1) - Q_{n,1}^{(i)}(\pm 1)| \le Cn^{-2s+2i}\omega_{\varphi^\lambda}^r\big(f^{(s)}, n^{-2+\lambda}\big)$$

for $0 \le i \le s$, and induction imply

$$|\alpha_i| + |\beta_i| \le C_1 n^{-2s+2i} \omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-2+\lambda}\big) \quad \text{for} \quad 0 \le i \le s. \tag{4.3.11}$$

We now use (4.2.12) (Trigub's Lemma) to write

$$\|(1-x^2)^\ell - (1-x^2)^{\ell+m} P_{n,\ell}(x)\| \le \frac{C(m,\ell)}{n^{2\ell}} \quad \ell, m = 1, 2, \dots . \tag{4.3.12}$$

We construct

$$Q_{n,3}(x) = \sum_{i=1}^{s} (1-x^2)^{i+m} \big(\alpha_i(1-x) + \beta_i(1+x)\big) P_{n,i}(x) \tag{4.3.13}$$

with some m, $m \ge s+1$. Showing that $P_n(x)$, given by

$$P_n(x) = Q_{n,0}(x) + Q_{n,1}(x) + Q_{n,2}(x) - Q_{n,3}(x),$$

satisfies both (4.1.4) and (4.1.5) when q_n (which defines $Q_{n,0}$) is given by (4.3.1) and both (4.1.6) and (4.1.7) when q_n is given by (4.3.2) will complete the proof. (Slight modifications are needed for (4.1.7) and $\ell = 1$ i.e. $\omega^1_{\varphi^\lambda}$ should replace $\omega^2_{\varphi^\lambda}$ in (4.3.2)).

We now prove Theorem 4.1.1. Using (4.3.11) and (4.3.12), we have

$$\|Q_{n,3}(x) - Q_{n,2}(x)\| \le C n^{-2s} \omega^r_{\varphi^\lambda}(f^{(s)}, n^{-2+\lambda}). \tag{4.3.14}$$

Using the Markov inequality, we deduce from (4.3.14)

$$\|Q^{(i)}_{n,3}(x) - Q^{(i)}_{n,2}(x)\| \le C n^{-2s+2i} \omega^r_{\varphi^\lambda}\big(f^{(s)}, n^{-2+\lambda}\big). \tag{4.3.15}$$

For $-1+n^{-2} \le x \le 1-n^{-2}$ (4.1.4) is valid. For $-1 \le x \le -1+n^{-2}$ (or $1-n^{-2} \le x \le 1$) we use $f^{(i)}(-1) - P_n^{(i)}(-1) = 0$ for $0 \le i \le s$ (or $f^{(i)}(+1) - P_n^{(i)}(+1) = 0$) and the Taylor formula to obtain

$$f^{(i)}(x) - P_n^{(i)}(x) = \frac{(1+x)^{s-i}}{(s-i)!}\left(f^{(s)}(\xi) - P_n^{(s)}(\xi)\right) \tag{4.3.16}$$

$$\text{for} \quad -1 \le \xi < x \le -1+n^{-2}.$$

Hence, (4.3.7) and (4.3.15) for $i = s$ imply

$$|f^{(i)}(x) - P_n^{(i)}(x)| \le C(1+x)^{s-i}\omega_{\varphi^\lambda}^r\left(f^{(s)}, n^{-2+\lambda}\right)$$

$$\text{for} \quad -1 \le x \le -1+n^{-2}$$

which in turn implies (4.1.4) as $(1+x) \le \varphi(x)/n$ for $-1 \le x \le -1+n^{-2}$. Using the Markov-Bernstein inequality, that is,

$$|P_n^{(i)}(x)| \le Cn^i \delta_n(x)^{-i} \|P_n\|_{C[-1,1]}, \quad P_n \in \Pi_n$$

we now deduce from (4.3.14) the inequality

$$|Q_{n,3}^{(i)}(x) - Q_{n,2}^{(i)}(x)| \le Cn^{-2s+i}\delta_n(x)^{-i}\omega_{\varphi^\lambda}^r(f^{(s)}, n^{-2+\lambda}).$$

This inequality together with (4.3.8) implies that we have to prove (4.1.5) only for $Q_{n,0}^{(s+k)} = q_n^{(k)}$. For $k = r$ we recall that the estimate of $q_n^{(r)}$ is given by Theorem 2.6.3 in Chapter 2 (or see Theorem 6.3 of [5]), that is, we have

$$|q_n^{(r)}(x)| \le Cn^r \delta_n(x)^{-r}\omega_{\varphi^\lambda}^r(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}). \tag{4.3.17}$$

The estimate of $|q_n^{(k)}(x)|$ for $k > r$ follows the same method. For $k > r$ we use the Bernstein and the Markov inequalities on the intervals $[\frac{-1-x}{2}, \frac{1+x}{2}]$ observing that $\delta_n(u)^{-m}\omega_{\varphi^\lambda}^r(f^{(s)}, n^{-1}\delta_n(u)^{1-\lambda})$ is bounded by $C(m)\delta_n(x)^{-m}\omega_{\varphi^\lambda}^r(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda})$ for u in that interval.

For the proof of (4.1.6) we may use (4.3.2) and hence $f^{(s)}(\pm 1) = q_n(\pm 1)$. Furthermore, $Q_{n,3}^{(s)}(\pm 1) = 0$ and therefore (4.3.10) reduces to

$$Q_{n,1}^{(s)}(\pm 1) + Q_{n,2}^{(s)}(\pm 1) = 0. \tag{4.3.18}$$

We can now prove the inequality (4.1.6). Note that for $-1 + n^{-2} < x < 1 - n^{-2}$ (4.1.6) has already been proved as in this case, there is no difference between (4.1.4) and (4.1.6) but the constant. For $-1 \le x \le -1 + n^{-2}$ (or $1 - n^{-2} < x \le 1$) we use both (4.3.16) and (4.3.17) and write

$$\begin{aligned} f^{(i)}(x) - P_n^{(i)}(x) &= \frac{(1+x)^{s-i}}{(s-i)!}\left(f^{(s)}(\xi) - P_n^{(s)}(\xi)\right) \\ &= \frac{(1+x)^{s-i}}{(s-i)!}\left(f^{(s)}(\xi) - q_n(\xi)\right) + \frac{(1+x)^{s-i}}{(s-i)!}(1+\xi) \\ &\qquad \times \left(-Q_{n,1}^{(s+1)}(\eta) - Q_{n,2}^{(s+1)}(\eta) + Q_{n,3}^{(s+1)}(\eta)\right) \\ &= I_1 + I_2, \end{aligned}$$

for $-1 < \eta < \xi < x \le -1 + n^{-2}$. Using (4.3.2) and $\varphi(\xi) \le \varphi(x)$, we conclude that I_1 is smaller than the right hand side of (4.1.6). Using (4.3.8) for $k = 1$

and (4.3.15) for $i = s+1$, we have

$$|I_2| \le C(1+x)^{s-i+1} n^2 \omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-2+\lambda}\big)$$

$$\le C\big(n^{-1}\varphi(x)\big)^{s-i}(1+x) n^2 \omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-2+\lambda}\big).$$

We now note that for any F, $\omega^r_{\varphi^\lambda}(F,t)$ satisfies

$$\omega^2_{\varphi^\lambda}(F,t) \le C_1 a^{-2} \omega^2_{\varphi^\lambda}(F, at)$$

for $a < 1$. Setting $F = f^{(s)}$, $t = n^{-2+\lambda}$ and $a = \big(n\varphi(x)\big)^{1-\lambda}$, we have

$$|I_2| \le C_1\big(n^{-1}\varphi(x)\big)^{s-i}(1+x) n^2 \big(n\varphi(x)\big)^{-2+2\lambda} \omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\big)$$

$$\le C_1\big(n^{-1}\varphi(x)\big)^{s-i} \omega^2_{\varphi^\lambda}\big(f^{(s)}, n^{-1}\varphi(x)^{1-\lambda}\big).$$

This completes the proof of (4.1.6).The estimate (4.1.7) follows that of Theorem 6.3 of [5] almost word for word. □

References

[1] Ju. A. Brudnyĭ, *Generalizations of a theorem of A. F. Timan*, Dokl. Akad. Nauk SSSR **148** (1963), 1237–1240 (Russian); Soviet Math. Dokl. **4** (1963), 244–247 (English).

[2] R. A. DeVore, *Pointwise approximation by polynomials and splines*, Theory of approximation of functions (S. B. Stečkin and S. A. Teljakovskiĭ, eds.), (proceedings of an Intern. Conf. Kaluga 1975), Nauka Moskow, 1977, pp. 132–141.

[3] R. A. DeVore, *Degree of approximation*, Approximation Theory II (G. G. Lorentz, C. K. Chui and L. L. Schumaker, eds.), (Proc. Int. Sympos. Austin TX 1976), Academic Press NY, 1976, pp. 117–161.

[4] Z. Ditzian, *On interpolation of* $L_p[a,b]$ *and weighted Sobolev spaces*, Pacific J. Math. **90** 2, (1980) 307-323.

[5] Z. Ditzian and D. Jiang, *Approximation by polynomials in* $C[-1,1]$, Canad. J. Math. **44** 5, (1992) 924-940.

[6] Z. Ditzian, D. Jiang and D. Leviatan, *Shape preserving polynomial approximation in* $C[-1,1]$, Proc. Camb. Phil. Soc. **112** (1992) 309-316.

[7] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, 1987.

[8] L. O. Ferguson, "Approximation by Polynomials with Integral Coefficients," Amer. Math. Soc., Providence (1980).

[9] H. Gonska and E. Hinnemann, *Punktweise abschätzungen zur approximation durch algebraische polynome*, Acta Math. Hung. **46** (1985), 143–254.

[10] I. E. Gopengauz, *A theorem of A.F. Timan on the approximation of functions by polynomials on a finite segment*, Mat. Zametki (1967) 163-172 (Russian); Math. Notes **1** (1967) 110-116 (English).

[11] I. E. Gopengauz, *A question about the approximation of functions on a finite segment and in a domain with corners*, Teor. Funkcii Funkcional. Anal. i Pril. **4** (1967), 204–210 (Russian).

[12] T. Kilgore, *An elementary simultaneous approximation theorem*, Proc. Amer. Math. Soc. (to appear).

[13] S. A. Teljakovskiĭ, *Two theorems on the approximation of functions by algebraic polynomials*, Mat. Sb. **70** (1966), 252–265 (Russian).

[14] A. F. Timan, *Strengthening of Jackson's theorem on the approximation of continuous functions on a finite segment of the real axis*, Dokl. Akad. Nauk SSSR **78** (1951), 17–20 (Russian).

[15] R.M. Trigub, *Approximation of functions by polynomials with integral coeffi-*

cients, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1962), 261-280 (Russian).

The results of this chapter are joint work with Z. Ditzian and D. Leviatan and will appear in SIAM Journal on Mathematical Analysis.

CHAPTER V.

APPROXIMATION BY POLYNOMIALS WITH INTERPOLATORY CONDITIONS

5.1. Introduction

For a certain class of smooth functions, it was shown that there exists a sequence of polynomials which satisfy certain inequalities. In chapters 2 and 4, we have studied approximation by real polynomials without any restrictive conditions. It is shown here that in one case we can require the polynomials to satisfy an interpolation condition at some given points and still satisfy the same inequalities.

With the new modulus of smoothness introduced by Ditzian and Totik [1],

$$\omega_\psi^r(f,t) \equiv \sup_{0<h\le t} \|\Delta_{h\psi}^r f\|,$$

where

$$\Delta_{h\psi}^r f(x) = \begin{cases} \sum_{j=0}^r (-1)^j \binom{r}{j} f(x + (j - \frac{r}{2})h\psi(x)) & \text{if } a \le x \pm \frac{r}{2}h \le b, \\ 0 & \text{otherwise,} \end{cases}$$

the following theorem of approximation by real polynomials is proved (Theorem 4.1.1

in Chapter 4):

Theorem 5.1.1. *Let* $f \in C^s[-1,1]$, $0 \le \lambda \le 1$ *and* $\varphi(x) = \sqrt{1-x^2}$. *There exists a sequence* $\{p_n\}$ *of the real polynomials, each of degree* n, *satisfying for* $-1 \le x \le 1$

$$|f^{(\nu)}(x) - p_n^{(\nu)}(x)| \le C\ \left[n^{-1}(n^{-1}+\varphi(x))\right]^{s-\nu} \times \omega_{\varphi^\lambda}^r\left(f^{(s)}, n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}\right),\ 0 \le \nu \le s \tag{5.1.1}$$

and for $k \ge r$

$$\left|p_n^{(s+k)}(x)\right| \le C\, n^k (n^{-1}+\varphi(x))^{-k} \omega_{\varphi^\lambda}^r\left(f^{(s)}, n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}\right), \tag{5.1.2}$$

where C *is independent of* f *and* n.

By making a change of variable, (5.1.1) remains valid on any finite interval $[a,b]$, $x \in [a,b]$ with $\varphi(x) = \sqrt{(x-a)(b-x)}$. From now on, we assume $\varphi(x) = \sqrt{(x-a)(b-x)}$ unless otherwise indicated.

In this chapter, we prove

Theorem 5.1.2. *Let* $f \in C^s[a,b]$, $0 \le \lambda \le 1$ *and let* $\{x_1, \cdots, x_l\}$ *be some given points in* $[a,b]$. *There exists a sequence* $\{P_n\}$ *of real polynomials of degree* $\le n$, $n \ge l(2s+r+1)$, *satisfying (5.1.1) and (5.1.2) such that*

$$P_n^{(\nu)}(x_j) = f^{(\nu)}(x_j),\ 1 \le j \le l,\ 0 \le \nu \le s. \tag{5.1.3}$$

5.2. Some Lemmas

In this section we state the following known lemmas and their corollaries which will be required later.

Lemma 5.2.1. ([1]) *For any function* $f \in C[a,b]$ *and* $t,\ \mu > 0$, *there exists a constant* C *independent of* f, t *and* μ *such that*

$$\omega^r_{\varphi^\lambda}(f, \mu t) \le C(1+\mu^r)\omega^r_{\varphi^\lambda}(f,t).$$

Lemma 5.2.2. *Suppose that* Ψ *is a nondecreasing nonnegative function satisfying for some constant* C

$$\Psi(\mu t) \le C(1+\mu^r)\Psi(t), \quad \mu, t > 0. \tag{5.2.1}$$

Suppose that for $m \ge 0$ *and* $0 \le \lambda \le 1$, *a polynomial* $p_n(x)$ *of degree* n *satisfies for* $\varphi(x) = \sqrt{1-x^2}$

$$|p_n(x)| \le C_1 \left[n^{-1}(n^{-1}+\varphi(x))\right]^m \Psi\left(n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}\right)$$

$$-1 \le x \le 1 \tag{5.2.2}$$

Then, for any integer $\ell > 0$, *we have*

$$\left|p_n^{(\nu)}(x)\right| \le C_2 \left[n^{-1}(n^{-1}+\varphi(x))\right]^{m-\nu} \Psi\left(n^{-1}(n^{-1}+\varphi(x))^{1-\lambda}\right),$$

$$-1 \le x \le 1,\ \ 0 \le \nu \le \ell, \tag{5.2.3}$$

where C_2 *is independent of* n *and* x.

Proof: See Lemma 4.2.1 in Chapter 4. □

Lemma 5.2.3. ([2]) *For any three positive integers* $\ell < m < n$ *there exist real numbers* a_j *and* b_j ($m \le j \le n$) *such that*

$$\left\| x^\ell - \sum_{j=m}^{n} a_j x^j \right\|_{C[-\gamma,\gamma]} \le C\frac{\gamma^\ell}{n^\ell} \tag{5.2.4}$$

and

$$\left\| x^\ell - \sum_{j=m}^{n} b_j x^j \right\|_{C[0,\gamma]} \le C\frac{\gamma^\ell}{n^{2\ell}} \tag{5.2.5}$$

where C *is independent of* n *and* γ.

Corollary 5.2.4. *Let* $\{x_1, \cdots, x_l\} \subset [a,b]$ *and let* $x_1 = a$, $x_l = b$. *If* $\gamma = b-a$, *and* $1 \le \ell = \nu \le s \le m = 2s+r+1 < n$ *in (5.2.5), then there exist real numbers* b_i, $m \le i \le n$, *such that for* $j = 1, l$

$$\left\| (x-x_j)^\nu - \sum_{i=2s+r+1}^{n} b_i (x-x_j)^i \right\|_{C[a,b]} \le C\frac{\gamma^\ell}{n^{2\ell}}.$$

Equivalently, there exists a polynomial $w_{n,j,\nu}(x)$ *such that*

$$\left\|(x-x_j)^\nu - (x-x_j)^{2s+r+1} w_{n,j,\nu}(x)\right\|_{C[a,b]} \le \frac{C_1}{n^{2\nu}}, \quad a \le x \le b. \tag{5.2.6}$$

The polynomials $w_{n,j,\nu}(x)$, $j = 1, \ell$, *are of degree* $\leq n - (2s + r + 1)$.

Similarly, applying (5.2.4) of Lemma 5.2.3 to $x - x_j$ for $j = 2, \cdots, l-1$, we have

Corollary 5.2.5. *Under the same hypothesis as in Corollary 5.2.4, there exist real numbers* a_i, $m \leq i \leq n$, *such that for* $1 \leq \nu \leq s$ *and* $j = 2, \cdots, l-1$

$$\|(x - x_j)^\nu - (x - x_j)^{2s+r+1} w_{n,j,\nu}(x)\|_{C[a,b]} \leq \frac{C}{n^\nu}, \quad a \leq x \leq b. \tag{5.2.7}$$

The polynomials $w_{n,j,\nu}(x)$ *are also of degree* $\leq n - (2s + r + 1)$.

5.3 Proof of the Theorem

The plan of the proof is to first start with the sequence of the polynomials $\{p_n\}$ guaranteed by Theorem 5.1.1. Secondly, we construct the polynomials $u_n(x)$ which satisfy certain Hermite interpolatory conditions on the set $\{x_1, \cdots, x_l\}$. (We assume that $x_1 = a$ and $x_l = b$.) Then we construct the auxiliary polynomials $v_n(x)$ which are related to $u_n(x)$ and satisfy

$$v_n^{(\nu)}(x_j) = 0, \quad 1 \leq j \leq l, \quad 0 \leq \nu \leq s. \tag{5.3.1}$$

The required polynomials $P_n(x)$ which will satisfy (5.1.1), (5.1.2) and (5.1.3) are shown to be given by

$$P_n(x) \equiv u_n(x) - v_n(x) + p_n(x). \tag{5.3.2}$$

The proof consists of three parts: 1) construction of $u_n(x)$, 2) construction of the auxiliary polynomial $v_n(x)$, and 3) finding estimate of $u_n(x) - v_n(x)$.

Construction of $u_n(x)$. Assume that $\{p_n\}$ is a sequence of polynomials satisfying (5.1.1) and (5.1.2) of Theorem 5.1.1 on $[a, b]$. $u_n(x)$ will be required to satisfy the following interpolation conditions:

$$u_n^{(\nu)}(x_j) = \begin{cases} f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j) & ,\nu = 0, \cdots, s, \\ 0 & ,\nu = s+1, \cdots, 2s+r \end{cases} \qquad 1 \le j \le l \tag{5.3.3}$$

and we can write

$$u_n(x) = \sum_{j=1}^{l} \sum_{\nu=0}^{s} \left[f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j) \right] r_{j,\nu}(x) \tag{5.3.4}$$

where $\{r_{j,\nu}\}$ are the fundamental polynomials of degree $\le l(2s+r+1)-1$. Then $u_n(x)$ is of degree $\le l(2s+r+1)-1$. The polynomials $r_{j,\nu}(x)$ satisfy

$$r_{j,\nu}^{(\mu)}(x_i) = \delta_{j,i}\delta_{\nu,\mu}, \quad 1 \le i, j \le l, \quad 0 \le \nu \le s, \; 0 \le \mu \le 2s+r, \tag{5.3.5}$$

where $\delta_{i,j}$ is the Kronecker delta. More precisely

$$r_{j,\nu}(x) = (x - x_j)^{\nu} \left[\prod_{i=1, i \ne j}^{l} (x - x_i) \right]^{2s+r+1} m_{j,\nu}(x) \tag{5.3.6}$$

where $m_{j,\nu}(x) = C_{j,\nu}(1 + \sum_{k=1}^{2s+r+1-\nu} c_{j,k}(x - x_j)^k)$ of degree $\le 2s + r + 1 - \nu$ and $C_{j,\nu} = [\nu!(\prod_{i=1, i \ne j}^{l}(x_j - x_i))^{2s+r+1}]^{-1}$. The explicit values of $c_{j,k}$ are not easy to find and to the best of my knowledge are not known except in some special cases. But $\{c_{j,k}\}$ are independent of n.

When $j = 2, \cdots, l-1$, we can write

$$r_{j,\nu}(x) = (x - x_j)^{\nu} \left[-\varphi^2(x) \prod_{i=2, i\neq j}^{l-1} (x - x_i) \right]^{2s+r+1} m_{j,\nu}(x). \tag{5.3.7}$$

Auxiliary Polynomials $v_n(x)$. We now define an auxiliary sequence of polynomials $v_n(x)$ of degree $\leq n + l(2s + r + 1)$ and set

$$v_n(x) \equiv \left[-\varphi^2(x) \prod_{j=2}^{l-1} (x - x_j) \right]^{2s+r+1} \sum_{j=1}^{l} \sum_{\nu=1}^{s} \left[f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j) \right]$$

$$\times w_{n,j,\nu}(x)\, m_{j,\nu}(x). \tag{5.3.8}$$

where the polynomials $w_{n,j,\nu}(x)$ are given in Corollaries 5.2.4 and 5.2.5 respectively. It is easy to see from (5.3.8) that

$$v_n^{(\nu)}(x_j) = 0, \quad 1 \leq j \leq l, \quad 0 \leq \nu \leq s.$$

Therefore

$$(u_n - v_n)^{(\nu)}(x_j) = f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j), \quad 0 \leq \nu \leq s, \quad 1 \leq j \leq l.$$

So the polynomial $P_n(x) = u_n(x) - v_n(x) + p_n(x)$ of degree $\leq n + l(2s + r + 1)$ satisfies the interpolation conditions (5.1.3).

Estimate of $u_n(x) - v_n(x)$. We shall first prove

$$|u_n(x) - v_n(x)| \leq C[n^{-1}(n^{-1} + \varphi(x))]^s \omega_{\varphi^\lambda}^r (f^{(s)}, n^{-1}(n^{-1} + \varphi(x))^{1-\lambda}). \tag{5.3.9}$$

To estimate $u_n(x) - v_n(x)$ it is useful to first write $v_n(x)$ from (5.3.8) in the following form:

$$v_n(x) = \sum_{j=1,l}\sum_{\nu=1}^{s}\left[f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)\right]\left[(x-x_j)\prod_{i=1,i\neq j}^{l}(x-x_j)\right]^{2s+r+1}$$
$$\times w_{n,j,\nu}(x)\, m_{j,\nu}(x)$$
$$+\sum_{j=2}^{l-1}\sum_{\nu=1}^{s}\left[f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)\right]\left[-\varphi^2(x)(x-x_j)\prod_{i=2,i\neq j}^{l-1}(x-x_j)\right]^{2s+r+1}$$
$$\times w_{n,j,\nu}(x)\, m_{j,\nu}(x).$$

From the expressions (5.3.4) and (5.3.6), we have

$$u_n(x) - v_n(x) \equiv I_{1,n}(x) + I_{2,n}(x) + I_{3,n}(x) \tag{5.3.10}$$

where we set

$$I_{1,n}(x) \equiv \sum_{j=1}^{l}\left[f(x_j) - p_n(x_j)\right] r_{j,0}(x), \tag{5.3.11}$$

$$I_{2,n}(x) \equiv \sum_{j=1,l}\sum_{\nu=1}^{s}\left[f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)\right] d_{j,\nu,n}(x) \tag{5.3.12}$$

$$I_{3,n}(x) \equiv \sum_{j=2}^{l-1}\sum_{\nu=1}^{s}\left[f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)\right] e_{j,\nu,n}(x). \tag{5.3.13}$$

For brevity, we have put for $j = 1,\ l$

$$d_{j,\nu,n}(x) \equiv \left[(x - x_j)^\nu - (x - x_j)^{2s+r+1} w_{n,j,\nu}(x)\right]$$

$$\times \left[\prod_{i=1, i\neq j}^{l} (x - x_i)\right]^{2s+r+1} m_{j,\nu}(x), \tag{5.3.14}$$

and for $j = 2, \cdots, l-1$

$$e_{j,\nu,n}(x) \equiv \left[(x - x_j)^\nu - (x - x_j)^{2s+r+1}(x) w_{n,j,\nu}(x)\right]$$

$$\times \left[-\varphi^2(x) \prod_{i=2, i\neq j}^{l-1} (x - x_i)\right]^{2s+r+1} m_{j,\nu}(x). \tag{5.3.15}$$

Recalling Corollaries 5.2.4 and 5.2.5, we see that

$$\|d_{j,\nu,n}\|_{C[a,b]} \leq \frac{C}{n^{2\nu}} \quad \text{and} \quad |e_{j,\nu,n}(x)| \leq \frac{C}{n^\nu} \varphi^{2(2s+r+1)}(x). \tag{5.3.16}$$

Estimate for $I_{1,n}(x)$: For simplicity, denote $\delta_n(x) = n^{-1} + \varphi(x)$. From Theorem 5.1.1, the difference $|f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)|$ satisfies inequality (5.1.1). Since $r_{j,0}(x)$ are independent of n, the first and last terms in sum (5.3.11) for $j = 1, l$, satisfy (5.3.9) with some constant C. To estimate the remaining terms for $j = 2, \cdots, l-1$, from (5.3.7) we observe

$$|r_{j,\nu}(x)| \leq C\varphi^{2s+r+1}(x), \qquad a \leq x \leq b. \tag{5.3.17}$$

This gives

$$|f(x_j) - p_n(x_j)|\,|r_{j,\nu}(x)| \le C\left[n^{-1}\delta_n(x_j)\right]^s \times \omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-1}\delta_n(x_j)^{1-\lambda}\right)\varphi^{2(s+r+1)}(x).$$

Since by Lemma 5.2.1

$$\omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-1}\delta_n(x_j)^{1-\lambda}\right) \le C\omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\right) \times \left[1 + \left(\frac{\delta_n(x_j)^{1-\lambda}}{\delta_n(x)^{1-\lambda}}\right)^r\right], \tag{5.3.18}$$

it follows that $|f(x_j) - p_n(x_j)|\,|r_{j,\nu}(x)|$ satisfy (5.3.9) and so does $I_{1,n}(x)$.

Estimate for $I_{2,n}(x)$: Similarly for $j = 1, l$ and $\nu = 1, \cdots, s$, by using (5.3.16)

$$\left|f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)\right| |d_{j,\nu,n}(x)| \le Cn^{-2(s-\nu)}\omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-2+\lambda}\right) n^{-2\nu}$$
$$\le C\left[n^{-1}\delta_n(x)\right]^s \omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\right)$$

which verifies $I_{2,n}(x)$.

Estimate for $I_{3,n}(x)$: For $j = 2, \cdots, l-1$ and $\nu = 1, \cdots, s$ by (5.3.15), (5.3.16) and (5.3.18)

$$\left|f^{(\nu)}(x_j) - p_n^{(\nu)}(x_j)\right| |e_{j,\nu,n}(x)| \le C\left[n^{-1}\delta_n(x_j)\right]^{s-\nu} \times \omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-1}\delta_n(x_j)^{1-\lambda}\right) n^{-\nu}\varphi^{2(2s+r+1)}(x)$$
$$\le C\left[n^{-1}\delta_n(x)\right]^s \omega^r_{\varphi^\lambda}\left(f^{(s)}, n^{-1}\delta_n(x)^{1-\lambda}\right),$$

which completes the estimate of $I_{3,n}$. From (5.3.10) and the estimates for $I_{1,n}$, $I_{2,n}$ and $I_{3,n}$, we get (5.3.9).

To verify that $P_n(x)$ satisfy (5.1.1) and (5.1.2), it is enough to show that for $0 \le \nu \le s$ and $\nu = k + s$

$$|(u_n - v_n)^{(\nu)}(x)| \le C[n^{-1}(n^{-1} + \varphi(x))]^{s-\nu} \omega^r_{\varphi^\lambda}(f^{(s)}, n^{-1}(n^{-1} + \varphi(x))^{1-\lambda})$$

since $p_n(x)$ satisfies both (5.1.1) and (5.1.2).

By Lemma 5.2.1, $\omega^r_{\varphi^\lambda}(f^{(s)}, t)$ satisfies (5.2.1). Therefore, we can deduce the above inequalities by using Lemma 5.2.2 from (5.3.9).

Remark 5.3.1. *In the proof we made the assumption that both end points of the interval* $[a, b]$ *belong to the interpolation points* $\{x_1, \cdots, x_l\}$. *When this assumption is not satisfied, we may only add the one or both end points to the interpolation points. Then, the degree of* $P_n(x)$ *will be* $n + (l + 2)(2s + r + 1)$. *But the Hermite interpolation polynomial of degree* $\le l(2s+r+1)$ *satisfying (5.1.3) will also satisfy both (5.1.1) and (5.1.2) for* $l(2s + r + 1) \le n \le (l + 2)(2s + r + 1)$.

References

[1] Z. Ditzian and V. Totik, "Moduli of Smoothness," Springer-Verlag, 1987.

[2] L. O. Ferguson, "Approximation by Polynomials with Integral Coefficients," Amer. Math. Soc., Providence (1980).